Stochastic Framework for Symmetric Affine Matching between Point Sets

Sai Kit Yeung [1] and Pengcheng Shi [1,2]
[1] Department of Electronic and Computer Engineering,
Hong Kong University of Science and Technology, Kowloon, Hong Kong
[2] College of Biomedical Engineering, Southern Medical University, Guangzhou, China
{saikit@ust.hk, eeship@ust.hk }

Abstract

This paper presents a new approach to obtain symmetry in point matching problem. Here, symmetric matching means the essential property that the choices of source and target should not determine the eventual matching results. Most earlier approaches to achieve symmetric matching have been in deterministic fashions, where symmetry constraints are added into the matching cost functions to impose source-target symmetric property during the matching process. Nevertheless, these modified cost functions cannot generally converge to real ground truth, and further, the perfect *source-target symmetry cannot be achieved. Given initial forward and backward matching matrices pair, computed from any reasonable matching strategies, our approach yields perfectly symmetric mapping matrices from a stochastic framework that simultaneously considers the errors underneath the initial matching matrices and the imperfectness of the symmetry constraint. An iterative generalized total least square (GTLS) strategy has been developed such that perfect source-target symmetry is imposed.*

1. Introduction

Intuitively, the matching problem should be symmetric in nature if the matching pair is in continuous domain, i.e., the ground truth correspondences established between the two data inputs should be independent of their order to the matching process. However, the source-target symmetry cannot be achieved even in matching very simple object. One of the most common point matching algorithm, Iterative Closet Point (ICP) [1] is used to demonstrate this fact (Fig.1). Conventional matching algorithms adopt the term source (S) for the input data being transformed in the matching process, while the one keeping fixed for the source to match is referred to as target (T). In this paper, T_{12} and T_{21} are referred to the transformations solved by the forward and backward matching processes respectively.

Such asymmetric nature of the conventional approaches raises a question. If there are two different transformation results, which one should we take? Should we pick one and perform a simple inverse? It would not be an easy question as in real matching problem the ground truth is always unavailable. To solve this problem we must have a matching cost function which does not depend on the order of S and T, i.e., $E(S,T) = E(T,S)$ [5]. The following matching cost function is possible to achieve source-target symmetry:

$$E(S,T) = w \times E_{Sim}(S,T) + (1-w) \times E_{Sim}(T,S) \quad (1)$$

where E_{Sim} measures the similarity (e.g., image intensity and geometrical properties) between the data sets. The simplest case for the weighting factor w is 0.5.

Christensen [2] enforced an inverse consistency constraint with the similarity metric during the optimization process. In [2], the symmetric property in Eq. (1) was achieved, however, by separating the forward and reverse registration processes into two phases with an additional inverse consistency constraint:

$$E_f(S,T) = E_{Sim}(S,T) + \rho \times E_{Cons}(S,T) \quad (2)$$
$$E_r(T,S) = E_{Sim}(T,S) + \rho \times E_{Cons}(T,S) \quad (3)$$

$E_{Cons}(S,T)$ and $E_{Cons}(T,S)$ are equivalent to $||T_{12} - T_{21}^{-1}||$ and $||T_{21} - T_{12}^{-1}||$ respectively. Eq. (2) and (3) are alternately optimized until convergence. The main problem of Eq. (2) and (3) is that the consistency property is only part of the overall cost function. This formulation is only *asymptotically* inverse consistent when ρ tends to infinity.

All the above symmetric formulations yield a 1-to-1 symmetric mapping that is *deterministic* in nature and assuming both the observation (matching results) and the model (symmetric relation) are not perturbed by any errors:

$$T_{12} * T_{21} = I \quad (4)$$

However, unless you have a continuous representation of the object and also a precise transformation model, otherwise it would not be true in general. These imperfect transformation matrices mean that their underlying stochastic uncertainties should not be omitted.

The 18th International Conference on Pattern Recognition (ICPR'06)
0-7695-2521-0/06 $20.00 © 2006 IEEE

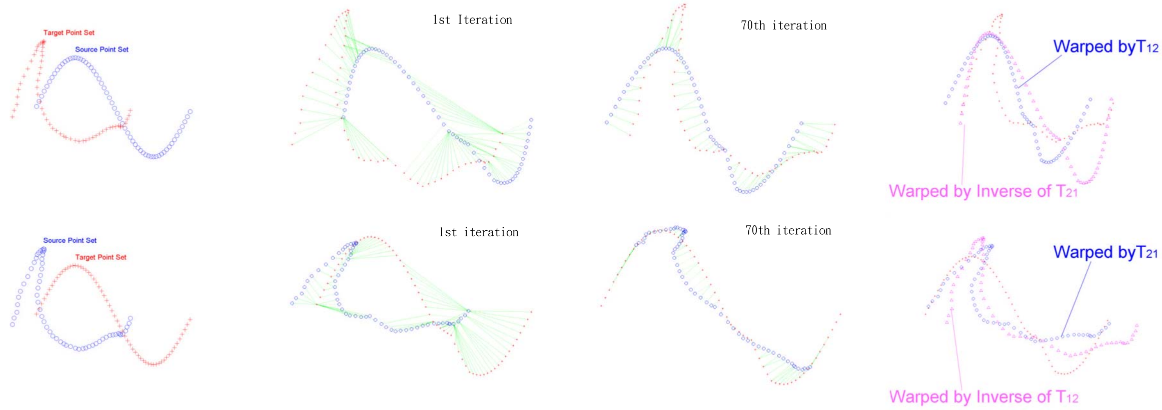

Figure 1. Forward and backward matching of 2 point sets to demonstrate inconsistent correspondences due to switch of the input order. Number of iterations in this ICP example is 149.

We propose a stochastic framework for matching problems which generates *perfect* source-target symmetric mapping between the data sets. Instead of imposing source-target symmetry in a *deterministic* and *asymptotic* sense, we enforce the symmetric property with the systematic considerations of the stochastic uncertainties of the input forward, backward transformation matrices and the symmetry constraint[1] to achieve *perfect* source-target symmetry. Given the forward and backward matching results, an iterative fitting process is performed until a set of new forward and backward transformation matrices are obtained which are *perfectly inverse to* each other. The fitting process is solved with the adoption of Generalized Total Least Square(GTLS) technique [6] which allows simultaneous considerations of all the errors in the input transformation matrices and the symmetry constraint. Here, we apply our stochastic framework to point matching problems and study the robustness of our system towards different amounts and types of noise on the input point sets. The framework can be applied to volumetric image registration problems with minor modification [7].

2. Stochastic framework for symmetric matching

2.1. Stochastic symmetry constraint

As we have stated above, the forward and backward transformation matrices obtained from any matching algorithm are going to be error-perturbed. Simply combine them deterministically may not be a good way to utilize the information from the forward and backward process. In this paper, we suggest that one should model the source-target symmetry stochastically with the simultaneous consideration of the underlying stochastic uncertainties within the forward and backward transformation matrices and hence the imperfectness of the symmetry constraint, i.e,

$$(T_{12}^G + E_{T_{12}}) * (T_{21}^G + E_{T_{21}}) = I + R \quad (5)$$

With Eq.(5), the observation and the model are no longer perfect but perturbed by noise. $E_{T_{12}}$ and $E_{T_{21}}$ are the errors associated with the ground truth transformations T_{12}^G and T_{21}^G. R is the error over identity due to 2 asymmetric matrices[2]. In our current error model, we assume all the elements in the error matrices are zero mean and are independent to each other.

2.2. Symmetry enforcement through iterative GTLS fitting

Since both the observation and the model are perturbed by noise, we adopted the Total Least Square (TLS) approach instead of the general Least Square (LS) to solve our problem. Also, as the stochastic property is not the same for every entry and some of the entries are error free, in order to solve the problem while considering all the errors simultaneously, a Generalized Total Least Square (GTLS) [6] approach is adopted.

The initial input of our framework are the transposed and permuted version of T_{12} and T_{21} (transformation matrices from the forward and backward matching process):

$$Q_{12}^{(0)} = T_{12}^T * P \qquad Q_{21}^{(0)} = T_{21}^T * P \quad (6)$$

$$invQ_{12}^{(0)} = (T_{12}^{-1})^T * P \qquad invQ_{21}^{(0)} = (T_{21}^{-1})^T * P \quad (7)$$

where $P = P_{14} * P_{24} * P_{34}$ and
$P_{14} = (e_4 e_2 e_3 e_1)$, $P_{24} = (e_1 e_4 e_3 e_2)$, $P_{34} = (e_1 e_2 e_4 e_3)$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \; e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \; e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \; e_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad (8)$$

[1]There are works about stochastic ICP, e.g., in [4] which impose the stochastic property on the point's location while we impose the stochastic property on the transformation matrices and also the symmetry constraint.

[2]We simply assume all the entries in R have the same stochastic uncertainty and set it as $\triangle_r$.

0-7695-2521-0/06 $20.00 © 2006 IEEE

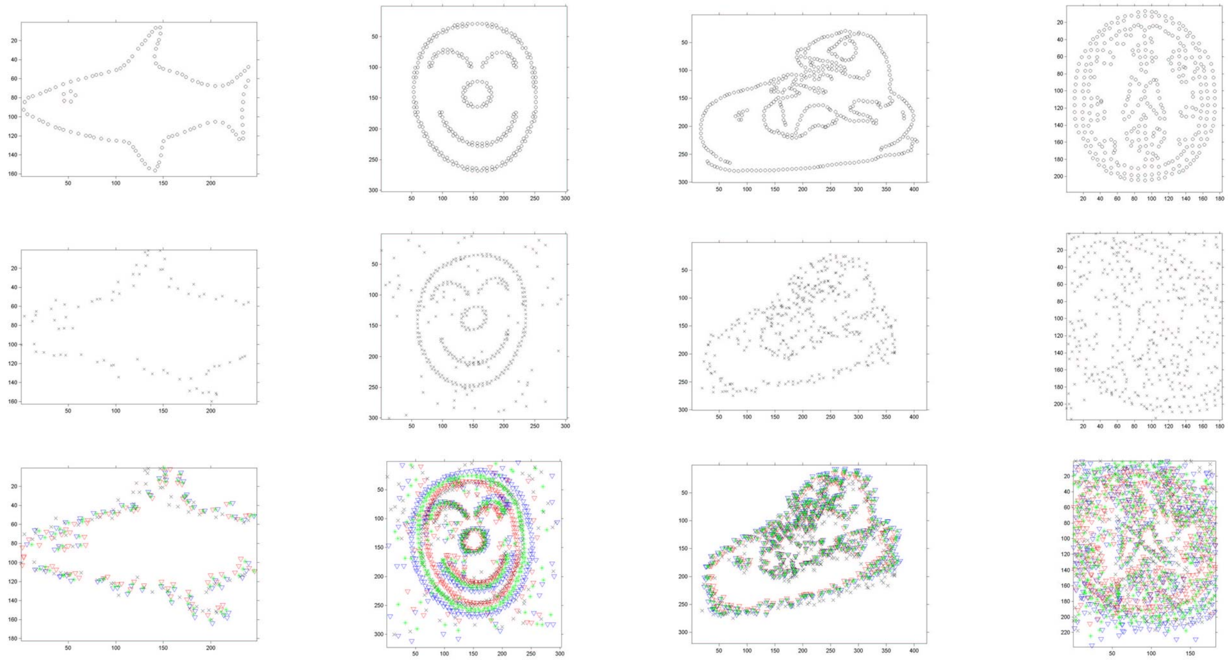

Figure 2. First row: different source point sets to the whole matching process. Second row: corresponding target point sets with different combinations of deformation and noise. Third row: the visual results of the forward process are shown. Different colors represent the source points warped by different transformations, red triangles(T_{12}), green stars(T^*_{12}), blue triangles(T^{-1}_{21}).

It is done because the first column of $Q^{(0)}_{12}$ and $Q^{(0)}_{21}$ will become error free. Our objective is to utilize the GTLS formulation to fit the transformation matrices iteratively, considering the errors in the transformation matrices and the source-target symmetry constraint until the forward and backward directions are symmetric:

$$\begin{bmatrix} Q^{(i)}_{12} \\ invQ^{(i)}_{21} \end{bmatrix} X \approx \begin{bmatrix} I \\ I \end{bmatrix} \quad \begin{bmatrix} invQ^{(i)}_{12} \\ Q^{(i)}_{21} \end{bmatrix} Y \approx \begin{bmatrix} I \\ I \end{bmatrix} \tag{9}$$

with the corresponding stochastic property in the noise data:

$$\begin{bmatrix} E^{(i)}_{Q_{12}} & R \\ E^{(i)}_{invQ_{21}} & R \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} E^{(i)}_{invQ_{12}} & R \\ E^{(i)}_{Q_{21}} & R \end{bmatrix} \tag{10}$$

where i is the iteration number. And

$$X^{-1} = invQ^{(i+1)}_{21} \ \text{ and } \ P * X * P = Q^{(i+1)}_{21} \tag{11}$$

$$Y^{-1} = invQ^{(i+1)}_{12} \ \text{ and } \ P * Y * P = Q^{(i+1)}_{12} \tag{12}$$

The process iterates until the degree of asymmetry (ϵ) is negligible:

$$\epsilon = ||(P * X)^T * (P * Y)^T - I||_F < \text{threshold} \tag{13}$$

and the GTLS solution matrices will be $T^*_{21} = (P * X)^T$, $T^*_{12} = (P * Y)^T$. The error matrices $E_{Q_{12}}$, $E_{invQ_{12}}$ for Q_{12}, $invQ_{12}$ are:

$$E^{(i)}_{Q_{12}} = |Q^{(i)}_{12} - invQ^{(i)}_{21}| \quad E^{(i)}_{invQ_{12}} = |Q^{(i)}_{21} - invQ^{(i)}_{12}| \tag{14}$$

i.e., absolute difference between 2 matrices which can be treated as the error's upper bound. And the first column of $E_{Q_{12}}$ is dropped as the first column of Q_{12} is error free. The error matrices $E_{invQ_{21}}$, $E_{Q_{21}}$ are formed respectively by:

$$E^{(i)}_{invQ_{21}} = \frac{(1-\alpha)}{\alpha} * E^{(i)}_{Q_{12}} \quad E^{(i)}_{Q_{21}} = \frac{(1-\alpha)}{\alpha} * E^{(i)}_{invQ_{12}} \tag{15}$$

where α is the weighting on the error of the forward matching process and can be picked by any prior knowledge of the input data or matching process. Therefore, α mimics weighting factor w in Eq.(1) in a stochastic sense. To simplify the model α is set to 0.5 in this paper. The error matrix R for the source-target symmetry constraint is fixed as the initial input stochastic consistent model is kept unchanged.

The error equilibration matrices for solving X in Eq.(9) are obtained from the Cholesky decomposition of the error covariance matrices C and D, where $C = \Delta^T \Delta$, $D = \Delta \Delta^T$,$\Delta = \begin{bmatrix} E_{Q_{12}} & R \\ E_{invQ_{21}} & R \end{bmatrix}$. Similarly Y is solved.

3. Experiments and conclusion

We have applied our stochastic symmetric model on different point sets. There are four different point sets in Fig.2. Two transformations, one is smaller deformed and one is larger deformed, are applied on each of the point set to form the corresponding target point set. Then, different amounts of gaussian noise and impulse noise are separately added to both the source and target point sets in the experiment. The forward and backward transformation matrices for our system inputs are obtained by the Robust point matching algorithm RPM [3]. Results evaluation is based on the sum of squared distance (SSD) between the points in the warped source point set and the target point set. Fig.2 shows examples with different degree of deformation and combination of noise. The corresponding SSD is shown in Table.1, the order of SSD is 10^4 and 10^6 for the gaussian and impulse noise respectively. Notice that the error should be compared within the same direction of the transformation, i.e., (T_{12}, GTLS T_{12}, T_{21}^{-1}) in the forward direction, (T_{21}, GTLS T_{21}, T_{12}^{-1}) in the backward direction. The SSD of GTLS T_{12} is colored in green if it outperforms both $T_{12}$and T_{21}^{-1}, similarly for GTLS T_{21}. Notice that as shown in the table, even T_{12} gives a better result in the forward direction, T_{12}^{-1} may indeed give you a much worse result in the backward direction. Our GTLS approach yields a unique mapping with reasonable results in both directions.

We presented a novel framework for modelling the source-target symmetry stochastically, by simultaneously considering the stochastic uncertainties on both of the transformation matrices and the symmetry constraint through the Generalized Total Least square fitting from the transformation matrices obtained after the matching process. With our stochastic consistency model, symmetry property can be imposed perfectly with the consideration of any other similarity constraints. This work is supported in part by Hong Kong Research Grants Council CERG-HKUST6252/04E and by China 973 Program (2003CB716103).

References

[1] P. J. Besl and N. D. McKay. A method for registration of 3-d shapes. *IEEE Trans. Pattern Anal. Mach. Intell.*, 14(2):239–256, 1992.

[2] G. E. Christensen and H. J. Johnson. Consistent image registration. *IEEE Trans. Med. Imaging*, 20(7):568–582, 2001.

[3] H. Chui and A. Rangarajan. A new algorithm for non-rigid point matching. In *CVPR*, pages 2044–2051, 2000.

[4] R. S. J. Estépar, A. Brun, and C.-F. Westin. Robust generalized total least squares iterative closest point registration. In *MICCAI (1)*, pages 234–241, 2004.

[5] O. M. Skrinjar and H. Tagare. Symmetric, transitive, geometric deformation and intensity variation invariant nonrigid image registration. In *ISBI*, pages 920–923, 2004.

[6] S. Van Huffel and J. Vandewalle. Analysis and properties of the generalized total least squares problem $AX \approx B$ when some or all columns in A are subject to error. *SIAM J. Matrix. Anal. Appl.*, 10:294–315, 1989.

[7] S. Yeung and P. Shi. Stochastic inverse consistency in medical image registration. In *MICCAI*, 2005.

Position Errors (SSD) - Fish										
	Small Deformation									
	Gaussian Noise(S.D)					Impulse Noise (Proportion)				
Transformation	1	2	3	4	5	0.1	0.2	0.3	0.4	0.5
T_{12}	143	100	130	270	173	24	26	51	61	91
GTLS T_{12}	145	99	125	283	175	29	32	61	74	98
T_{21}^{-1}	153	107	135	330	206	48	46	77	95	110
T_{21}	216	142	187	408	228	30	31	51	69	94
GTLS T_{21}	226	146	197	419	218	28	34	55	72	104
T_{12}^{-1}	260	171	244	520	260	35	48	68	79	126
	Large Deformation									
	Gaussian Noise(S.D)					Impulse Noise (Proportion)				
Transformation	1	2	3	4	5	0.1	0.2	0.3	0.4	0.5
T_{12}	178	218	241	285	538	39	45	52	63	123
GTLS T_{12}	187	233	250	285	615	39	48	55	67	104
T_{21}^{-1}	197	252	277	346	728	40	56	86	71	102
T_{21}	94	124	144	168	366	210	271	328	441	583
GTLS T_{21}	91	119	135	153	346	217	290	370	465	597
T_{12}^{-1}	91	122	144	178	364	237	354	654	519	756

Position Errors (SSD) - Smile face										
	Small Deformation									
	Gaussian Noise(S.D)					Impulse Noise (Proportion)				
Transformation	1	2	3	4	5	0.1	0.2	0.3	0.4	0.5
T_{12}	11	41	101	164	259	56	148	205	268	373
GTLS T_{12}	10	41	89	161	271	89	169	224	292	405
T_{21}^{-1}	12	46	96	177	331	197	201	250	319	434
T_{21}	14	57	114	213	375	86	165	228	252	382
GTLS T_{21}	13	51	110	201	327	72	177	247	265	401
T_{12}^{-1}	15	53	134	217	346	81	205	286	291	436
	Large Deformation									
	Gaussian Noise(S.D)					Impulse Noise (Proportion)				
Transformation	1	2	3	4	5	0.1	0.2	0.3	0.4	0.5
T_{12}	15	65	153	230	436	121	271	266	389	504
GTLS T_{12}	16	60	134	236	479	112	231	269	411	554
T_{21}^{-1}	22	70	141	283	597	129	249	320	481	647
T_{21}	12	42	91	165	333	77	150	204	301	393
GTLS T_{21}	9	37	87	147	292	89	208	251	346	454
T_{12}^{-1}	10	41	101	156	311	137	447	386	458	583

Position Errors (SSD) - Raptor										
	Small Deformation									
	Gaussian Noise(S.D)					Impulse Noise (Proportion)				
Transformation	1	2	3	4	5	0.1	0.2	0.3	0.4	0.5
T_{12}	42	44	54	51	72	123	353	521	584	811
GTLS T_{12}	465	48	57	53	81	134	407	565	648	887
T_{21}^{-1}	50	51	62	57	92	144	474	605	713	957
T_{21}	67	69	83	73	115	144	379	564	647	861
GTLS T_{21}	66	66	80	71	108	147	401	587	685	905
T_{12}^{-1}	65	65	81	75	110	156	451	631	756	986
	Large Deformation									
	Gaussian Noise(S.D)					Impulse Noise (Proportion)				
Transformation	1	2	3	4	5	0.1	0.2	0.3	0.4	0.5
T_{12}	194	185	215	214	251	20	38	65	88	114
GTLS T_{12}	192	183	214	208	266	20	37	62	97	122
T_{21}^{-1}	191	183	216	211	283	21	42	69	110	130
T_{21}	97	95	112	110	155	14	26	44	68	89
GTLS T_{21}	99	96	113	112	147	13	28	52	72	95
T_{12}^{-1}	103	99	117	122	143	14	36	81	81	106

Position Errors (SSD) - Brain										
	Small Deformation									
	Gaussian Noise(S.D)					Impulse Noise (Proportion)				
Transformation	1	2	3	4	5	0.1	0.2	0.3	0.4	0.5
T_{12}	4	9	18	23	89	57	86	127	168	236
GTLS T_{12}	3	7	14	23	94	61	97	142	187	256
T_{21}^{-1}	4	8	15	27	103	66	111	157	205	272
T_{21}	5	11	19	33	124	40	96	134	185	240
GTLS T_{21}	4	10	19	30	122	42	98	141	194	249
T_{12}^{-1}	7	13	27	33	132	46	107	156	211	266
	Large Deformation									
	Gaussian Noise(S.D)					Impulse Noise (Proportion)				
Transformation	1	2	3	4	5	0.1	0.2	0.3	0.4	0.5
T_{12}	45	53	67	57	99	58	99	148	221	257
GTLS T_{12}	49	54	70	58	104	61	96	160	225	280
T_{21}^{-1}	54	57	77	64	115	66	111	174	243	313
T_{21}	234	268	397	326	591	40	70	113	168	204
GTLS T_{21}	232	286	420	320	592	42	86	118	186	222
T_{12}^{-1}	255	348	524	363	671	46	131	130	229	256

Table 1. Sum of squared distances (SSD) of different data sets under different combinations of transformation, deformation and noise. Gaussian noise with Standard.Deviation = 1,2,3,4,5 and Impulse noise with proportion = 0.1,0.2,0.3,0.4,0.5 to the number of original points. Transformations for forward direction: (T_{12}, GTLS T_{12}, T_{21}^{-1}), backward direction: (T_{21}, GTLS T_{21}, T_{12}^{-1}). This table is color-coded.