Philippe Flajolet, Divide & Conquer Recurrences and The Mellin-Perron Formula

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The Problem

Most basic Divide-and-Conquer Recurrence is in form

$$f(n) = 2f\left(\frac{n}{2}\right) + e_n \qquad f_1,\ e_n \text{ given and } n \geq 2$$

Well known that if

$$e_n = \begin{cases} o(n) & \Rightarrow & f_n & = & \Theta(n) \\ \Theta(n) & \Rightarrow & f_n & = & \Theta(n \log n) \\ \Theta(n^k),\ k > 1 & \Rightarrow & f_n & = & \Theta(n^k) \end{cases}$$

What's left to do?

The Problem

Not so simple. When *n* is odd, set can't be split into two *equal* subsets. Use *almost equal* subsets. Recurrence becomes

$$f_n = f_{\lfloor n/2 \rfloor} + f_{\lceil n/2 \rceil} + e_n$$

- Solutions can get quite complicated
  Second order and sometimes even first order terms can be functions periodic in *lg n*

- These periodic functions can be complicated.
  Usually continuous, sometimes not differentiable.

- Same periodicity phenomenon occurs in some arithmetic functions.

- Previously, deriving periodic functions was ad-hoc and time consuming

- As a master of techniques, Philippe realized that Mellin-transform methods were applicable. He showed how they provided an "elementary" derivation

Two Basic Examples

- Worst case number of comparisons used by recursive Mergesort when sorting *n* items

- Total number of 1's in binary representation of integers less than *n*

Example 1: Worst Case Mergesort

Worst Case # of comparisons used by recursive mergesort on *n* items.

$$\forall n \geq 2, \quad f_n = f_{\lfloor n/2 \rfloor} + f_{\lceil n/2 \rceil} + n - 1; \qquad f_1 = 0$$

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Solution is

$$f_n = n \lg n + nA(\lg n) + 1$$

{x} is fractional part of x,
e.g., {2.7} =2

lg n ≡ log$_2$ n

where $A(u) = 1 - \{u\} - 2^{1-\{u\}}$

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A(x) is periodic with period 1, i.e., *A(x+1) = A(x),*
and continuous, with *A(0) = A(1).*

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Very old analysis.
Appears in Knuth, Vol 1

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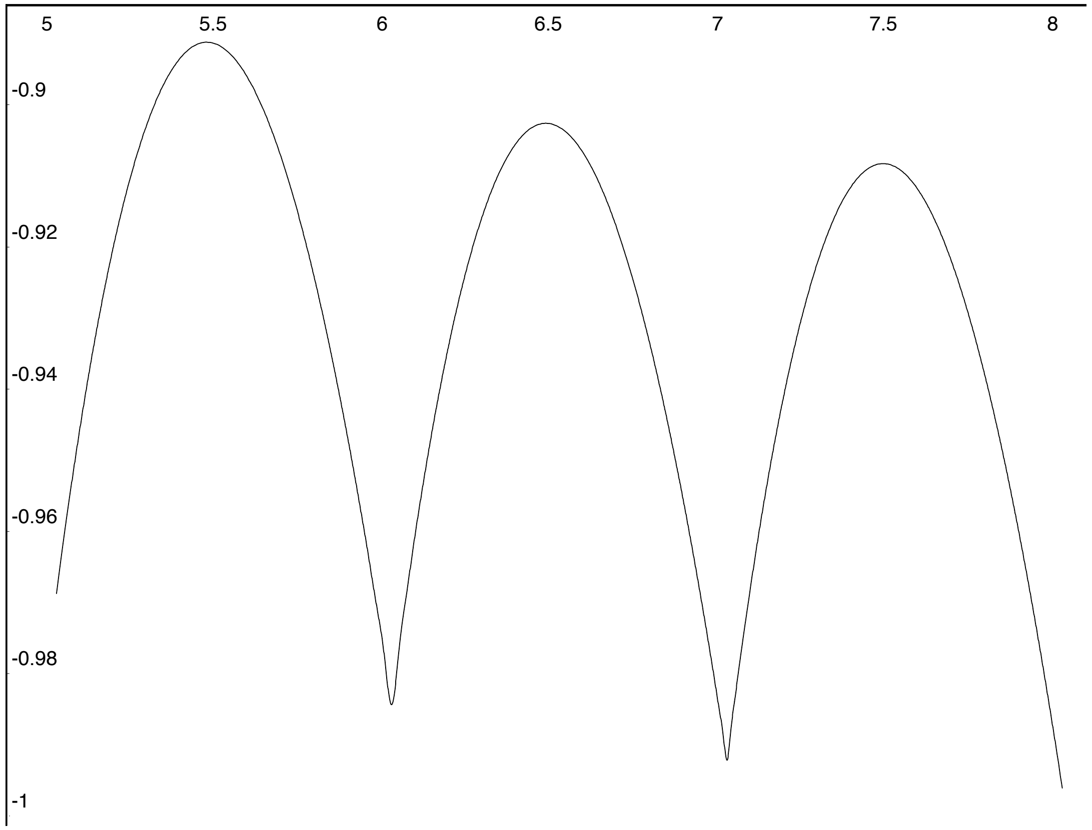

$$f_n = f_{\lfloor n/2 \rfloor} + f_{\lceil n/2 \rceil} + n - 1$$

$$\Rightarrow \quad f_n = n \lg n + nA(\lg n) + 1$$

Diagram shows $\frac{1}{n}(f_n - n \lg n)$

for $2^5 \leq n \leq 2^6$

Can see convergence to $A(u) = 1 - \{u\} - 2^{1-\{u\}}$

Example 2: Sum of Ones

Bin(n)	n	$v_1(n)$	$H(n)$
0	0	0	
01	1	1	0
10	2	1	1
11	3	2	2
100	4	1	4
101	5	2	5
110	6	2	7
111	7	3	9
1000	8	1	12
1001	9	2	13

$v_1(n)$ is # of 1's in binary representation of n.

$H(n) = \sum_{i<n} v_1(n)$ is an interesting arithmetic function.

Also arises in analysis of various algorithms.

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Delange (1975) proved (long, technical derivation) that

$$H(n) = \frac{1}{2} n \lg n + nD(\lg n)$$

where $D(n)$ is periodic of period 1, continuous,

but non-differentiable at points $\{\lg n\}$ for $n \in Z^+$.

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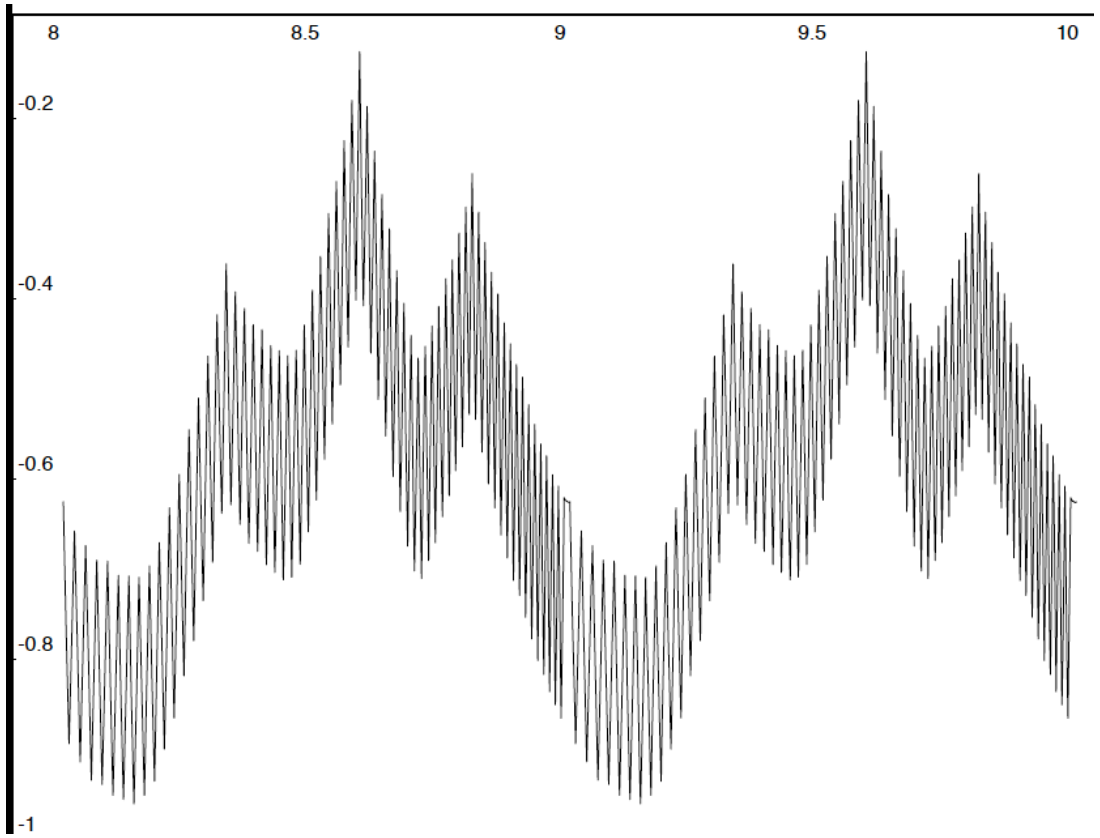

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Diagram graphs $\frac{1}{n}\left(H(n) - \frac{1}{2} n \lg n\right)$ for $2^8 \leq n \leq 2^{10}$.

Can see periodicity of $D(u)$.
$D(u)$ is continuous but not continuously differentiable

General Schema & Results

- Survey of Results
- Background to Technique
- General Schema

Some Results

For these and many similar problems, Mellin techniques can derive complete asymptotics. Some examples from refs are:

Problem	f_n definition	f_n solution
Worst case Mergesort	$f_{\lfloor n/2 \rfloor} + f_{\lceil n/2 \rceil} + n - 1$	$n \lg n + nA(\lg n) + 1$
Average Case Mergesort	$f_{\lfloor n/2 \rfloor} + f_{\lceil n/2 \rceil} + n - \gamma_n$	$n \lg n + nB(\lg n) + O(1)$
Variance of Mergesort	$f_{\lfloor n/2 \rfloor} + f_{\lceil n/2 \rceil} + \delta_n$	$n \cdot C(\lg n) + o(n)$
Sum of Digits Function	$\sum_{i<n} v_1(i)$	$\frac{1}{2} n \lg n + nD(\lg n)$
Triadic Binary Numbers	$\sum_{i<n} h(i)$	$n^{1+\lg 3} E(\lg n) - \frac{1}{4} n$
No of odd Binary Coeff in 1st n rows of Pascal's Triangle	$\sum_{i<n} 2^{v_1(i)}$	$n^{\lg 3} F(\lg n)$

$$\gamma_n = \frac{\lfloor n/2 \rfloor}{\lceil n/2 \rceil + 1} + \frac{\lceil n/2 \rceil}{\lfloor n/2 \rfloor + 1}$$

$$\delta_{2m+1} = \delta_{2m+2} = \frac{2m(m+1)^2}{(m+2)^2(m+3)}$$

$v_1(n)$ is # of "**1**"s in binary representation of n

$h(n)$ evaluates a base 2 number as a base 3 number

$$h\left(\sum_i 2^{e_i}\right) = \sum_i 3^{e_i} \qquad \text{e.g., } h(5) = h(101_2) = 3^2 + 1 = 10$$

For all of these problems, the *A(x)*, *B(x)*, etc., functions are continuous and periodic with period 1.

Mellin technique outputs function in Fourier Series form.

Basic Technique

Define *backward and forward differences* of sequences *f*, *g* by

$$\nabla f_n = f_n - f_{n-1} \qquad \Delta g_n = g_{n+1} - g_n$$

Double Difference is $$\Delta\nabla f_n = \nabla f_{n+1} - \nabla f_n$$

Easily seen that $$f(n) = n f_1 + \sum_{k=1}^{n-1} (n-k) \Delta\nabla f_k$$

Basic Technique II

The Mellin-Perron Formula (special case) states : if *c > 0* lies in the half-plane of absolute convergence of the Dirichlet generating function $W(s) = \sum_{n=1}^{\infty} \frac{w_n}{n^s}$

$$\Rightarrow \quad \sum_{k=1}^{n-1} (n-k) w_k = \frac{n}{2i\pi} \int_{c-i\infty}^{c+i\infty} W(s) n^s \frac{ds}{s(s+1)}$$

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In particular, set $w_n = \Delta\nabla f_n$ so $W(s) = \sum_{n=1}^{\infty} \frac{w_n}{n^s} = \sum_{n=1}^{\infty} \frac{\Delta\nabla f_n}{n^s}$

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Let $c>0$ lie in the half-plane of absolute convergence of $W(s)$. Then

$$\begin{aligned} f(n) &= n f_1 + \sum_{k=1}^{n-1} (n-k) \Delta\nabla f_k \\ &= n f_1 + \frac{n}{2i\pi} \int_{c-i\infty}^{c+i\infty} W(s) n^s \frac{ds}{s(s+1)} \end{aligned}$$

General Schema

(A) Set $w_n = \Delta\nabla f_n$ and calculate Dirichlet Generating Function

$$W(s) = \sum_{n=1}^{\infty} \frac{w_n}{n^s} = \sum_{n=1}^{\infty} \frac{\Delta\nabla f_n}{n^s}$$

Observation: For D & C Recurrences and arithmetic functions this can be *easy*.

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(B) Plug into Mellin-Perron formula and evaluate

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Observations: This usually reduces to computing residues.
Equally spaced residues along a vertical line yield periodic functions.

Divide & Conquer Recurrences

- When f_n defined by D&C Recurrence, its associated Dirichlet generating function has special form

- Fully Worked Example: Worst-Case Mergesort

- Another Example: Average-Case Mergesort

Divide & Conquer Recurrences

$$\forall n \geq 2, \quad f_n = f_{\lfloor n/2 \rfloor} + f_{\lceil n/2 \rceil} + e_n \qquad e_0 = f_0 = e_1 = 0$$

Splitting into odd and even cases yields

$$\begin{cases} f_{2m} & = & 2f_m + e_{2m} \\ f_{2m+1} & = & f_m + f_{m+1} + e_{2m+1} \end{cases} \Rightarrow \begin{cases} \nabla f_{2m} & = & \nabla f_m + \nabla e_{2m} \\ \nabla f_{2m+1} & = & \nabla f_{m+1} + \nabla e_{2m+1} \end{cases}$$

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$$\Rightarrow \begin{cases} \Delta\nabla f_{2m} &= \Delta\nabla f_m + \Delta\nabla e_{2m} \\ \Delta\nabla f_{2m+1} &= \Delta\nabla e_{2m+1} \end{cases}$$

for m > 0, with $\Delta\nabla f_1 = f_2 - 2f_1 = e_2 = \Delta\nabla e_1.$

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Setting $w_n = \Delta\nabla f_n$ and $W(s) = \sum_{n=1}^{\infty} \frac{w_n}{n^s}$ gives

$$W(s) = \sum_{m=1}^{\infty} \frac{\Delta\nabla f_m}{(2m)^s} + \Delta\nabla f_1 + \sum_{n=2}^{\infty} \frac{\Delta\nabla e_n}{n^s} = \frac{W(s)}{2^s} + \sum_{n=1}^{\infty} \frac{\Delta\nabla e_n}{n^s}.$$

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yielding $$W(s) = \frac{\Xi(s)}{1 - 2^{-s}} \quad \text{where} \quad \Xi(s) = \sum_{n=1}^{\infty} \frac{\Delta\nabla e_n}{n^s}.$$

Divide & Conquer Recurrences

$$\forall n \geq 2, \quad f_n = f_{\lfloor n/2 \rfloor} + f_{\lceil n/2 \rceil} + e_n \qquad\qquad e_0 = f_0 = e_1 = 0$$

We have just seen that *f* can be recovered via

$$f_n = nf_1 + \frac{n}{2i\pi} \int_{c-i\infty}^{c+i\infty} W(s) n^s \frac{ds}{s(s+1)}.$$

for

$$W(s) = \frac{\Xi(s)}{1 - 2^{-s}} \quad \text{where} \quad \Xi(s) = \sum_{n=1}^{\infty} \frac{\Delta \nabla e_n}{n^s}.$$

Observations: e_n are known so *W(s)* can be calculated.
If $e_n = O(n)$, then *W(s)* is absolutely convergent for *R(s)*>2,
and we may let *c=3*.

Simple Worked Example

$$f_1 = 0 \qquad \text{and} \qquad \forall n \geq 2,\ \ f_n = f_{\lfloor n/2 \rfloor} + f_{\lceil n/2 \rceil} + n - 1$$

$$\Rightarrow \quad e_n = n - 1 \quad \Rightarrow \quad \Delta\nabla e_1 = e_2 = 1 \quad \text{and} \quad \forall n \geq 2,\ \ \Delta\nabla e_n = 0.$$

This gives Dirichlet Generating Functions

$$\Xi(s) = \sum_{n=1}^{\infty} \frac{\Delta\nabla e_n}{n^s} = 1 \quad \text{and} \quad W(s) = \frac{\Xi(s)}{1 - 2^{-s}} = \frac{1}{1 - 2^{-s}}$$

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Plugging into Mellin-Perron Formula yields

$$f(n) = n f_1 + \frac{n}{2i\pi} \int_{3-i\infty}^{3+i\infty} W(s) n^s \frac{ds}{s(s+1)}$$

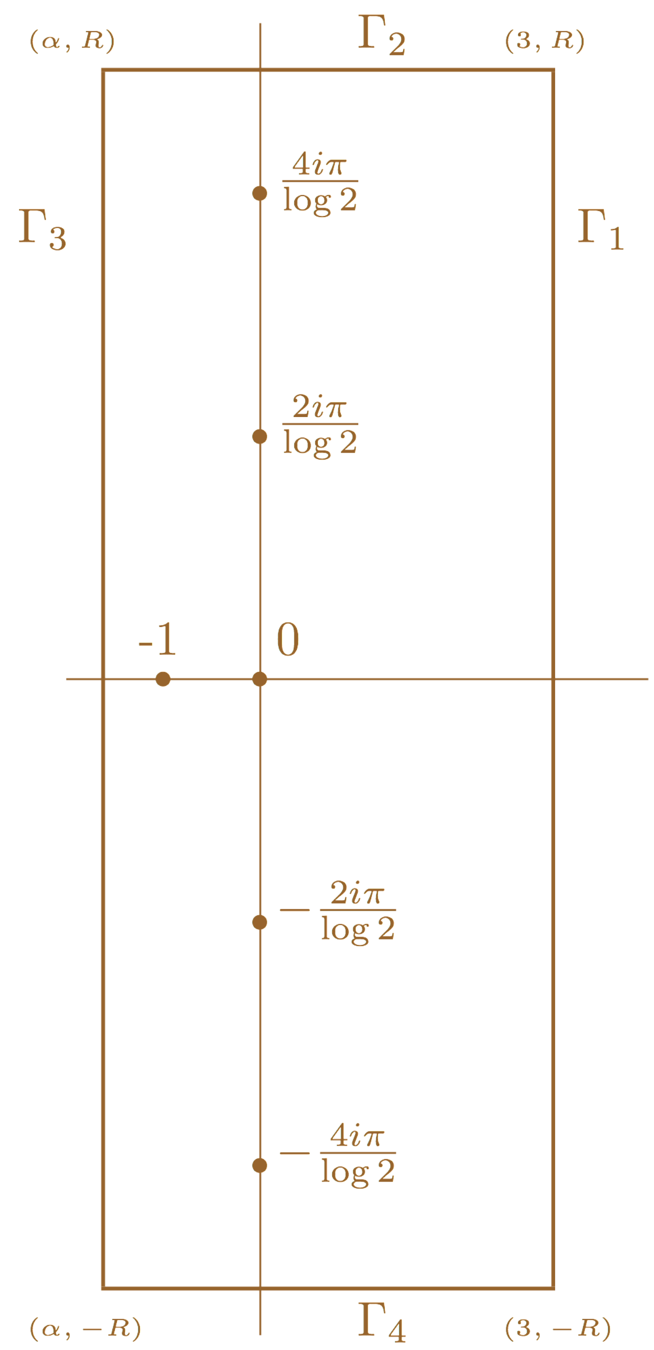
or

$$\boxed{\frac{f_n}{n} = \frac{1}{2i\pi} \int_{3-i\infty}^{3+i\infty} \frac{n^s}{1 - 2^{-s}} \frac{ds}{s(s+1)}}$$

Simple Worked Example (cont)

$$\frac{f_n}{n} = \frac{1}{2i\pi}\int_{3-i\infty}^{3+i\infty} I(s)\,ds \qquad \text{where} \quad I(s) = \frac{n^s}{1-2^{-s}}\,\frac{1}{s(s+1)}$$

Fix α<-1 and set R>0. Construct counterclockwise contour $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ and observe that



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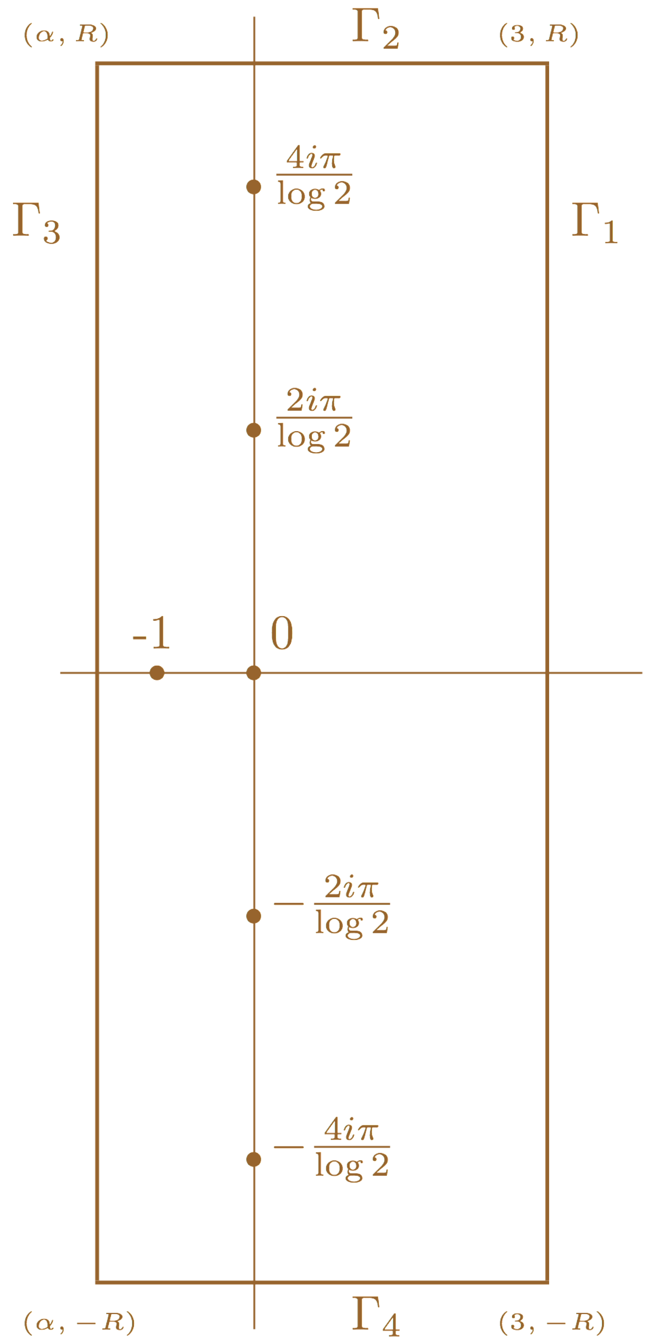

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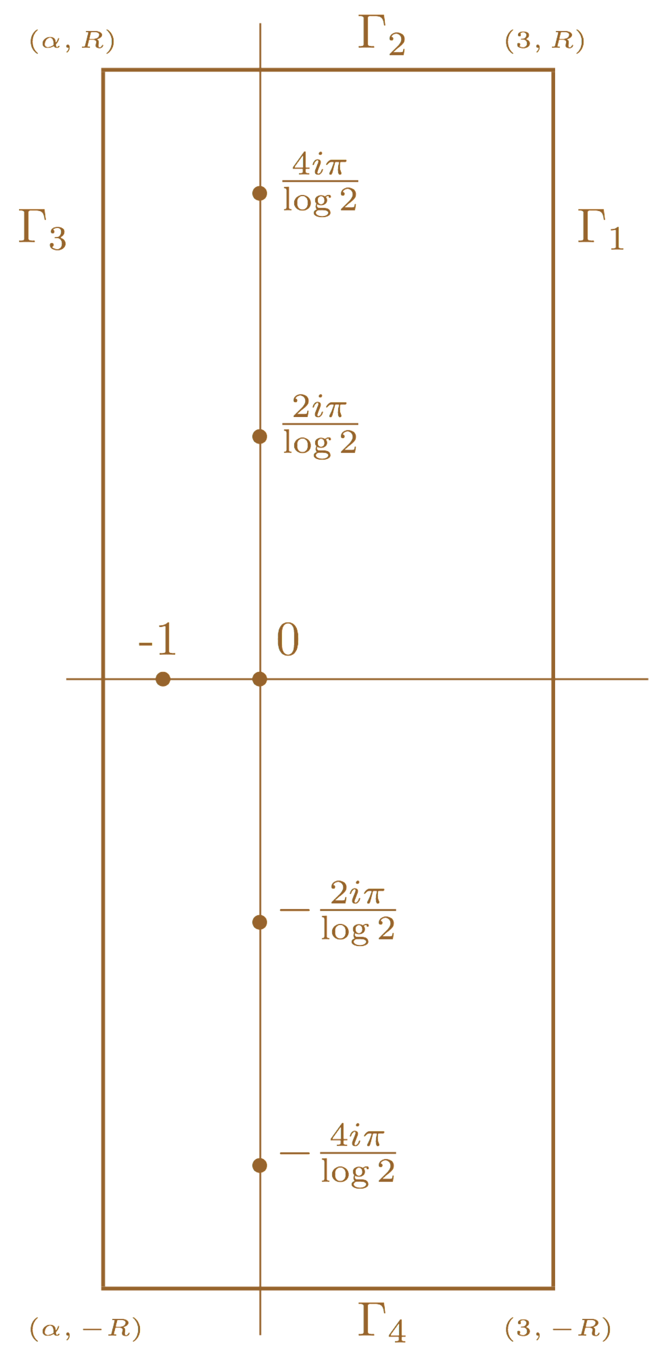

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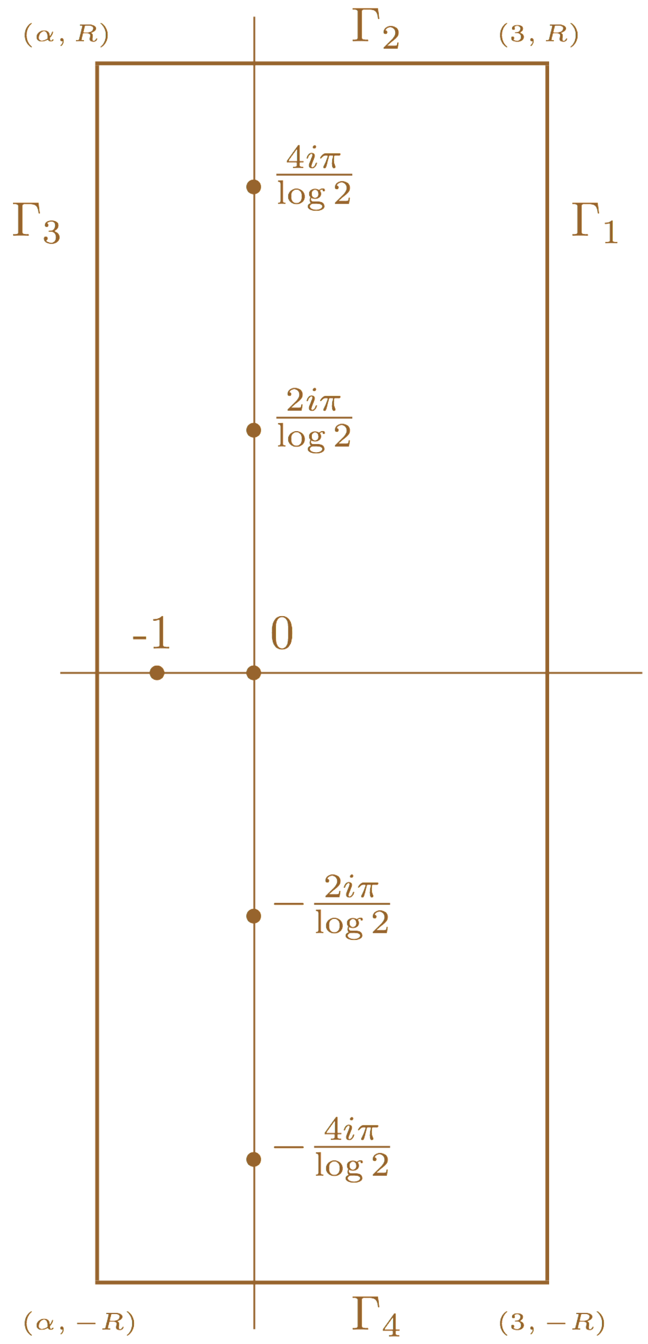

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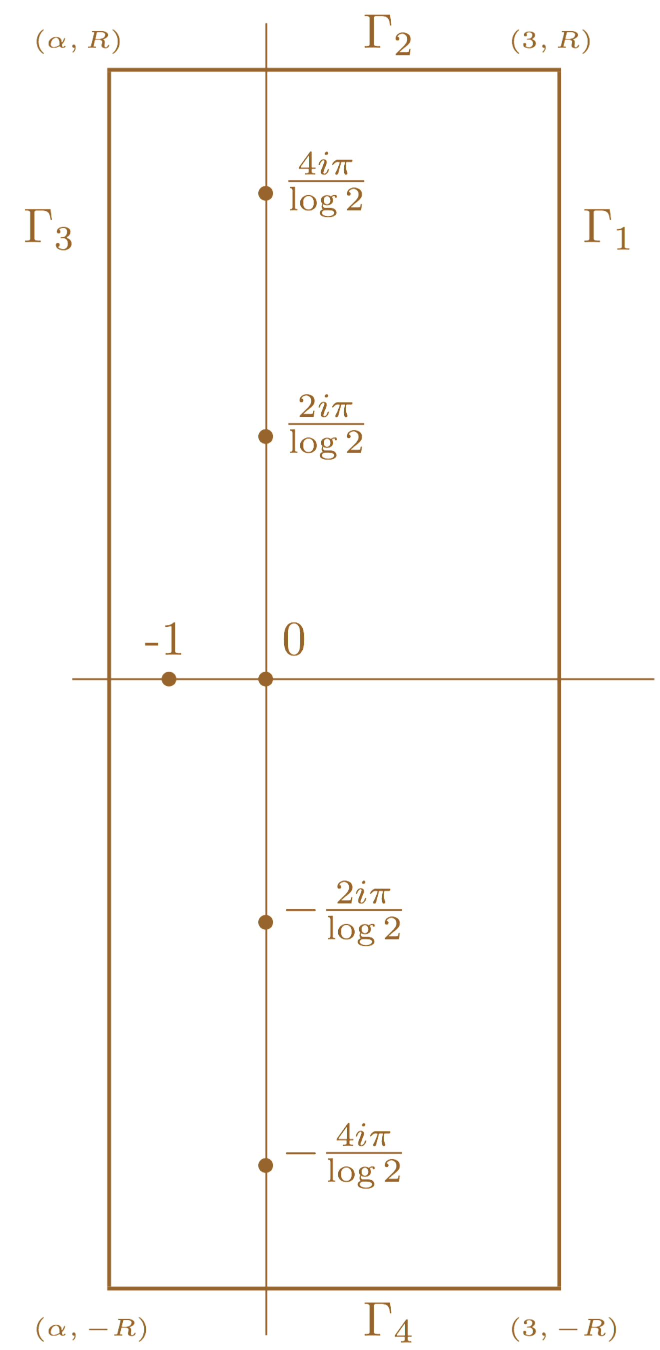
$$\lim_{R\to\infty}\frac{1}{2i\pi}\int_{\Gamma_1} I(s)ds = \frac{f_n}{n}$$

$$\Rightarrow \quad \frac{f_n}{n} = \lim_{R\to\infty}\frac{1}{2i\pi}\int_{\Gamma} I(s)ds + O(n^\alpha)$$



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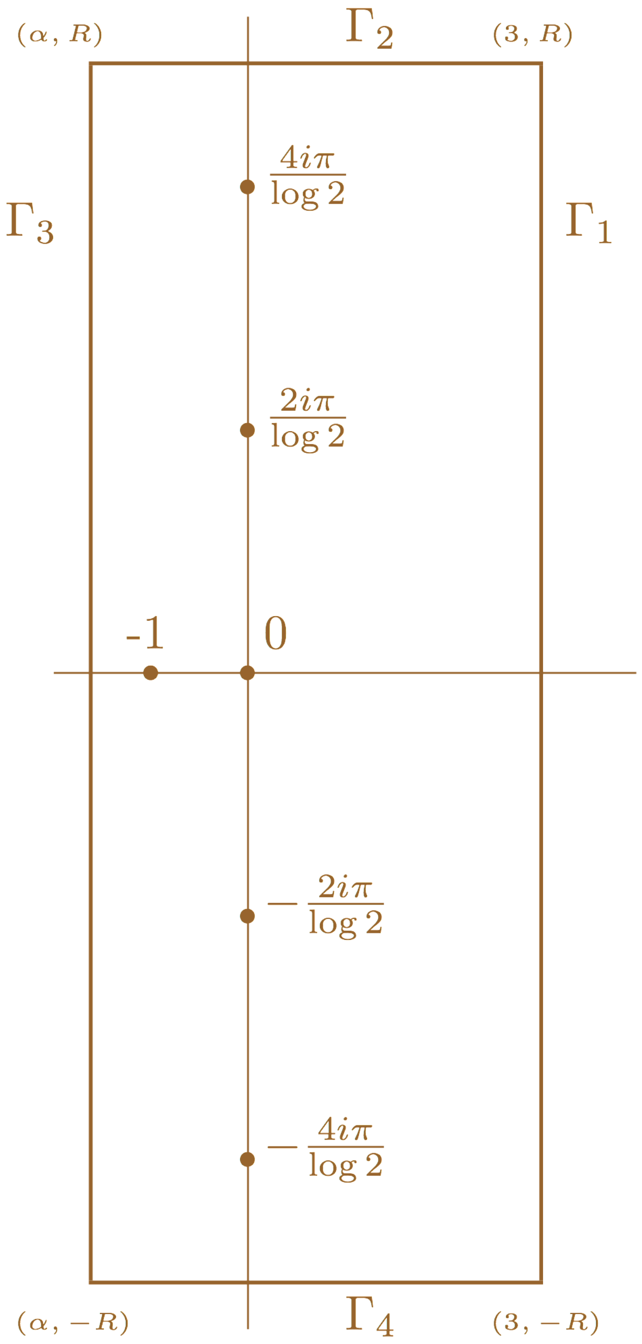
$$\lim_{R\to\infty}\frac{1}{2i\pi}\int_{\Gamma_2} I(s)ds = \lim_{R\to\infty}\frac{1}{2i\pi}\int_{\Gamma_4} I(s)ds = 0 \qquad \lim_{R\to\infty}\frac{1}{2i\pi}\int_{\Gamma_3} I(s)ds = O(n^{\alpha})$$

$$\lim_{R\to\infty}\frac{1}{2i\pi}\int_{\Gamma_1} I(s)ds = \frac{f_n}{n}$$

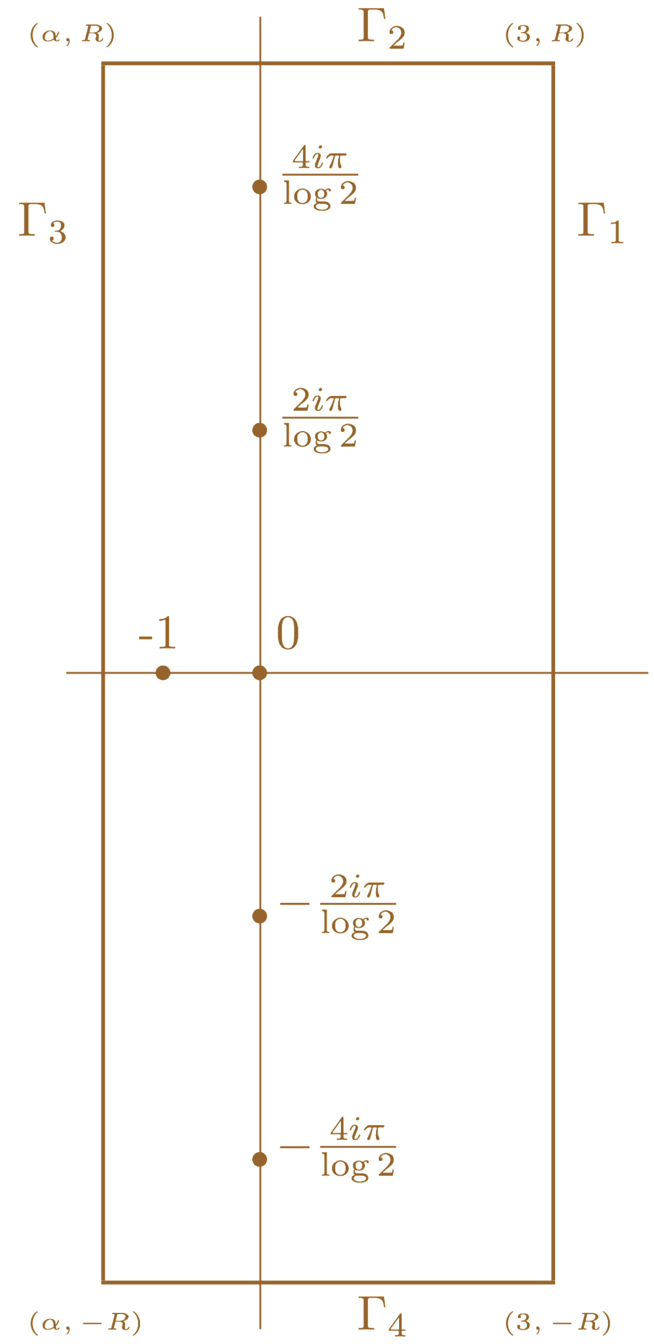
$$\Rightarrow \quad \frac{f_n}{n} = \lim_{R\to\infty}\frac{1}{2i\pi}\int_{\Gamma} I(s)ds + O(n^{\alpha})$$

This can be evaluated by adding up values of residues of I(s) within Γ!



Simple Worked Example (cont)

$$\frac{f_n}{n} = \frac{1}{2i\pi}\int_{3-i\infty}^{3+i\infty} I(s)\,ds \qquad \text{where} \quad I(s) = \frac{n^s}{1-2^{-s}}\,\frac{1}{s(s+1)}$$



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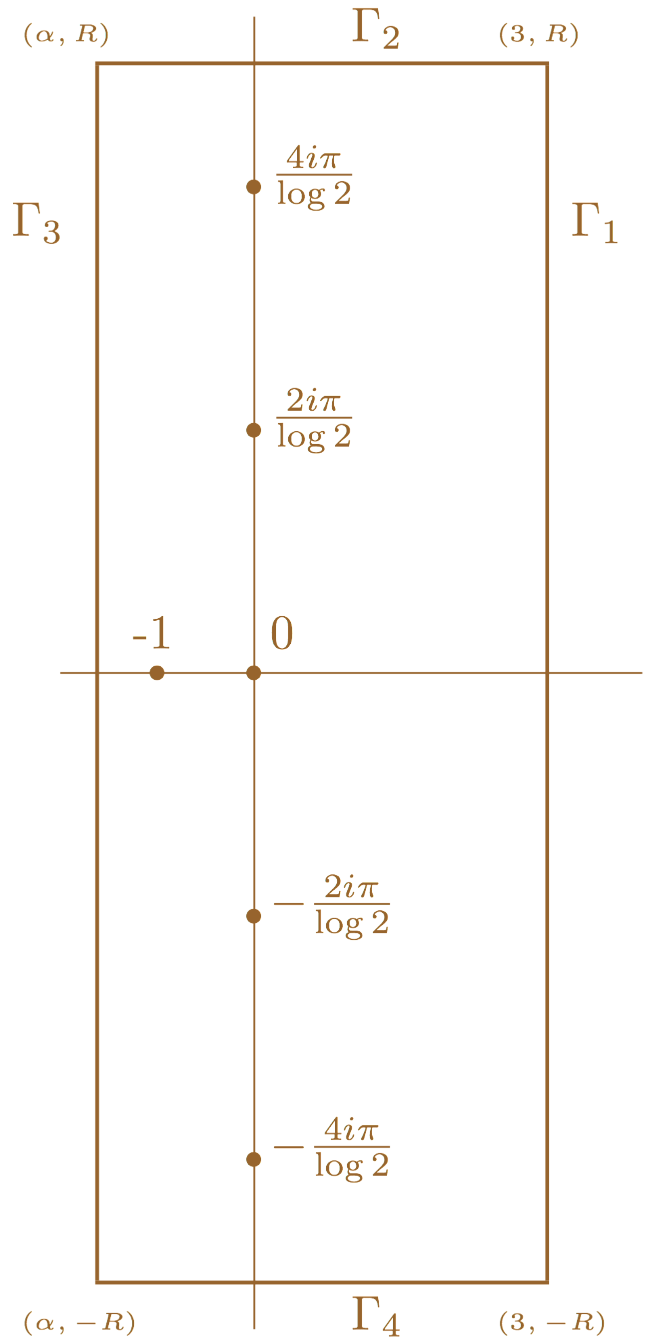
This can be evaluated by adding up values of residues of I(s) within Γ!

Note: This is true for all α<-1, so we actually get $\frac{f_n}{n} = \lim_{R\to\infty} \frac{1}{2i\pi}\int_{\Gamma} I(s)ds$

Simple Worked Example (cont)

$$\frac{f_n}{n} = \lim_{R\to\infty} \frac{1}{2i\pi} \int_\Gamma I(s)ds \qquad \text{where} \quad I(s) = \frac{n^s}{1-2^{-s}} \frac{1}{s(s+1)}$$



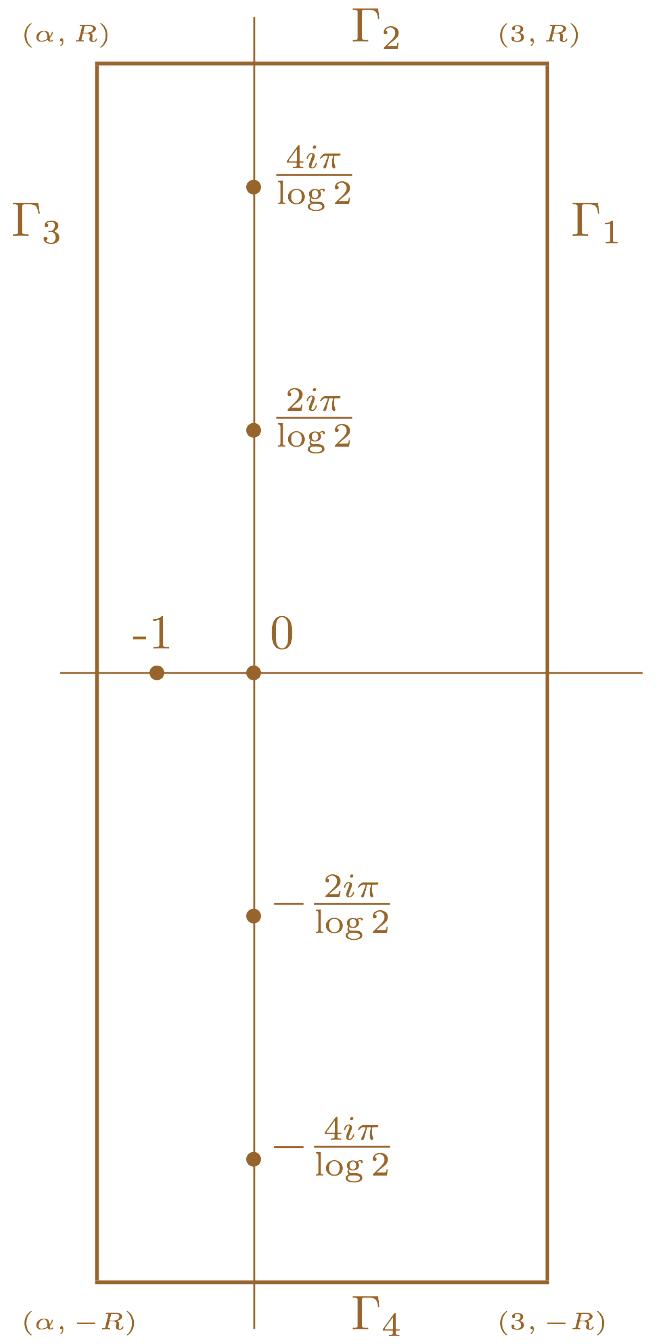
Simple Worked Example (cont)

$$\frac{f_n}{n} = \lim_{R\to\infty} \frac{1}{2i\pi} \int_\Gamma I(s)ds \qquad \text{where} \quad I(s) = \frac{n^s}{1-2^{-s}} \frac{1}{s(s+1)}$$

The singularities of $I(s)$ are

1. A double pole at $s = 0$ with residue $\lg n + \frac{1}{2} - \frac{1}{\log 2}$.
2. A simple pole at $s = -1$ with residue $\frac{1}{n}$.
3. Simple poles at $s = 2ki\pi/\log 2$, $k \in \mathbf{Z} \setminus \{0\}$ with residues $a_k e^{2ik\pi \lg n}$.

$$a_k = \frac{1}{\log 2} \frac{1}{\chi_k(\chi_k + 1)} \qquad \text{with} \quad \chi_k = \frac{2ik\pi}{\log 2}.$$



Simple Worked Example (cont)

$$\frac{f_n}{n} = \lim_{R\to\infty} \frac{1}{2i\pi}\int_\Gamma I(s)ds \qquad \text{where} \quad I(s) = \frac{n^s}{1-2^{-s}}\frac{1}{s(s+1)}$$

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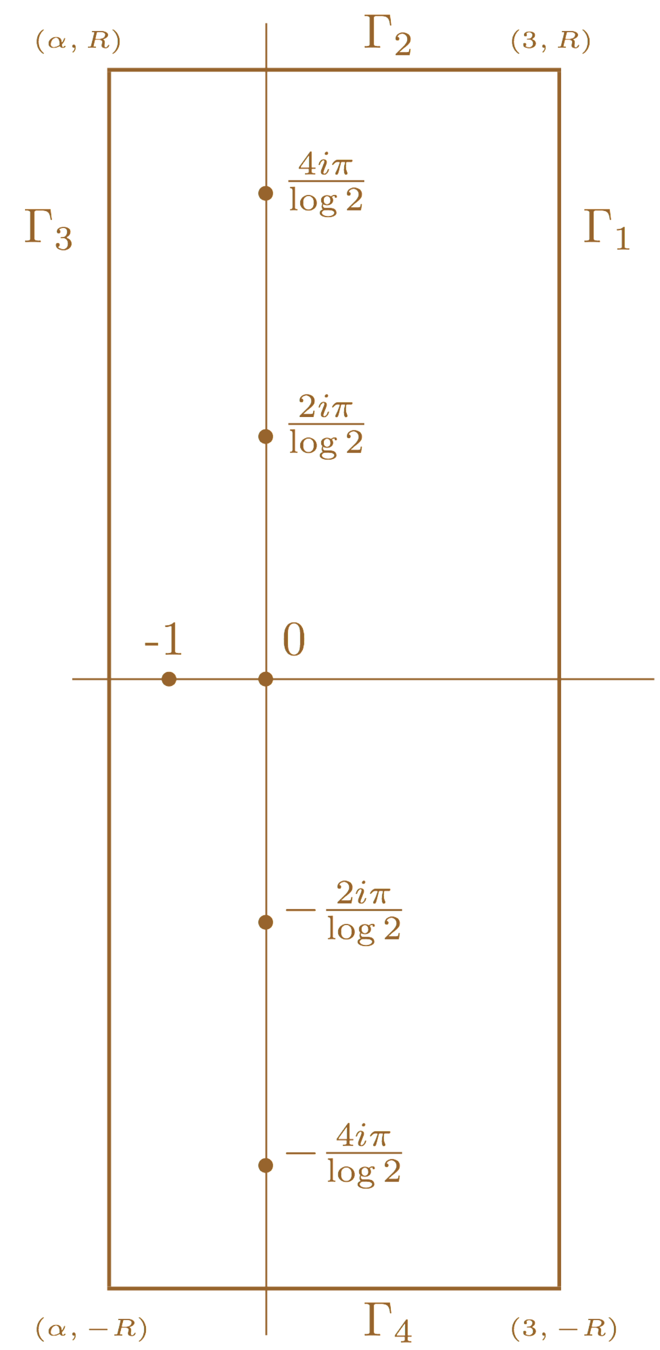

$$\Rightarrow \frac{f_n}{n} = n\lg n + nA(\lg n) + 1 + O(n^\alpha)$$

where $A(u)$ has explicit Fourier expansion

$$A(u) = \sum_{k\in\mathbf{Z}} a_k e^{2ik\pi u}, \quad \text{with} \quad a_0 = \frac{1}{2} - \frac{1}{\log 2}$$

Another Example: Average Case Mergesort

$$f_n = f_{\lfloor n/2 \rfloor} + f_{\lceil n/2 \rceil} + e_n \quad \text{where} \quad e_n = n - \left(\frac{\lfloor n/2 \rfloor}{\lceil n/2 \rceil + 1} + \frac{\lceil n/2 \rceil}{\lfloor n/2 \rfloor + 1} \right)$$

$$f_n = n \lg n + nB(\lg n) + O(1)$$

$B(u) = \sum_{k \in \mathbf{Z}} b_k e^{2ik\pi u}$ is given by uniformly convergent Fourier Series,

$$b_k = \frac{1}{\log 2} \frac{1 + \Psi(\chi_k)}{\chi_k(\chi_k + 1)} \qquad \text{with} \quad \chi_k = \frac{2ik\pi}{\log 2},$$

$$\Psi(s) = \sum_{m=1}^{\infty} \frac{2}{(m+1)(m+2)} \left[\frac{-1}{(2m)^s} + \frac{1}{(2m+1)^s} \right].$$

Note: *B(u)* is continuous, periodic with period 1, but non-differentiable at *all* values $\{\log_2 n\}$, for integers *n*.

Another Example: Average Case Mergesort

$$f_n = f_{\lfloor n/2 \rfloor} + f_{\lceil n/2 \rceil} + e_n \quad \text{where} \quad e_n = n - \left(\frac{\lfloor n/2 \rfloor}{\lceil n/2 \rceil + 1} + \frac{\lceil n/2 \rceil}{\lfloor n/2 \rfloor + 1} \right)$$

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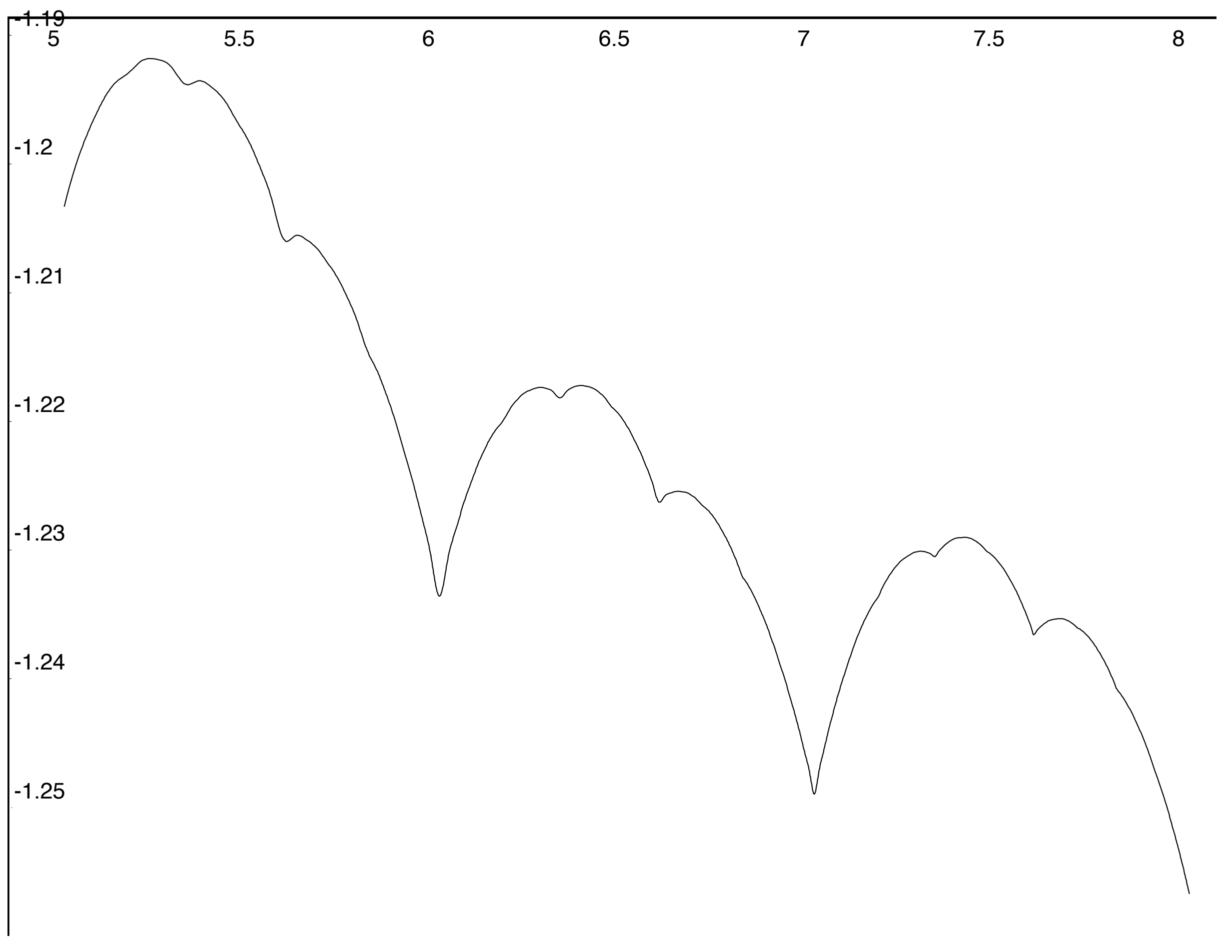

$(f_n - n \lg n)/n$ plotted for $2^5 \leq n \leq 2^6$ on logarithmic scale

Can see convergence to *B(lg n)*, periodicity and continuity of B(u), and non-differentiability of *B(u)*.

Arithmetic Functions

- Counting Number of 1's

  New Derivation; much “easier” than Delange (1975)

- Relationship with Reimann Zeta Function

Example: Sum of Digits

Bin(n)	n	$v_1(n)$	f_n	$v_2(n)$
0	0	0		
01	1	1	0	0
10	2	1	1	1
11	3	2	2	0
100	4	1	4	2
101	5	2	5	0
110	6	2	7	1
111	7	3	9	0
1000	8	1	12	3
1001	9	2	13	0

$v_1(n)$ is # of "**1**"s in binary representation of n

$$f_n = \sum_{n \geq 1} v_1(n)$$

$v_2(n)$ is exponent of 2 in prime decomposition of n

$$\nabla f_n = f_n - f_{n-1} = v_1(n-1)$$

$$\Delta\nabla f_n = \nabla f_{n+1} - \nabla f_n = v_1(n) - v_1(n-1) = 1 - v_2(n)$$

Recall the Reimann Zeta function $\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$

Known that $\sum_{n \geq 1} \frac{v_2(n)}{n^s} = \frac{\zeta(s)}{2^s - 1}$

Set $w_n = \Delta\nabla f_n$, $W(s) = \sum_{n=1}^{\infty} \frac{w_n}{n^s}$

$$\Rightarrow \quad W(s) = \sum_{n=1}^{\infty} \frac{1 - v_2(n)}{n^s} = \left[1 - \frac{1}{2^s - 1}\right] \zeta(s) = \frac{2^s}{1 - 2^s} \zeta(s)$$

Example: Sum of Digits

Bin(n)	n	$v_1(n)$	f_n	$v_2(n)$
0	0	0		
01	1	1	0	0
10	2	1	1	1
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110	6	2	7	1
111	7	3	9	0
1000	8	1	12	3
1001	9	2	13	0

$v_1(n)$ is # of "**1**"s in binary representation of n

$$f_n = \sum_{n \geq 1} v_1(n)$$

$$W(s) = \sum_{n=1}^{\infty} \frac{\Delta\nabla f_n}{n^s} = \frac{2^s}{1-2^s}\zeta(s)$$

M-P: $$f_n = \frac{n}{2i\pi}\int_{2-i\infty}^{2+i\infty} W(s)n^s \frac{ds}{s(s+1)}$$

Evaluating by integrating over appropriate contour and taking residues yields

$$f_n = \frac{1}{2} n \lg n + nF(\lg n)$$

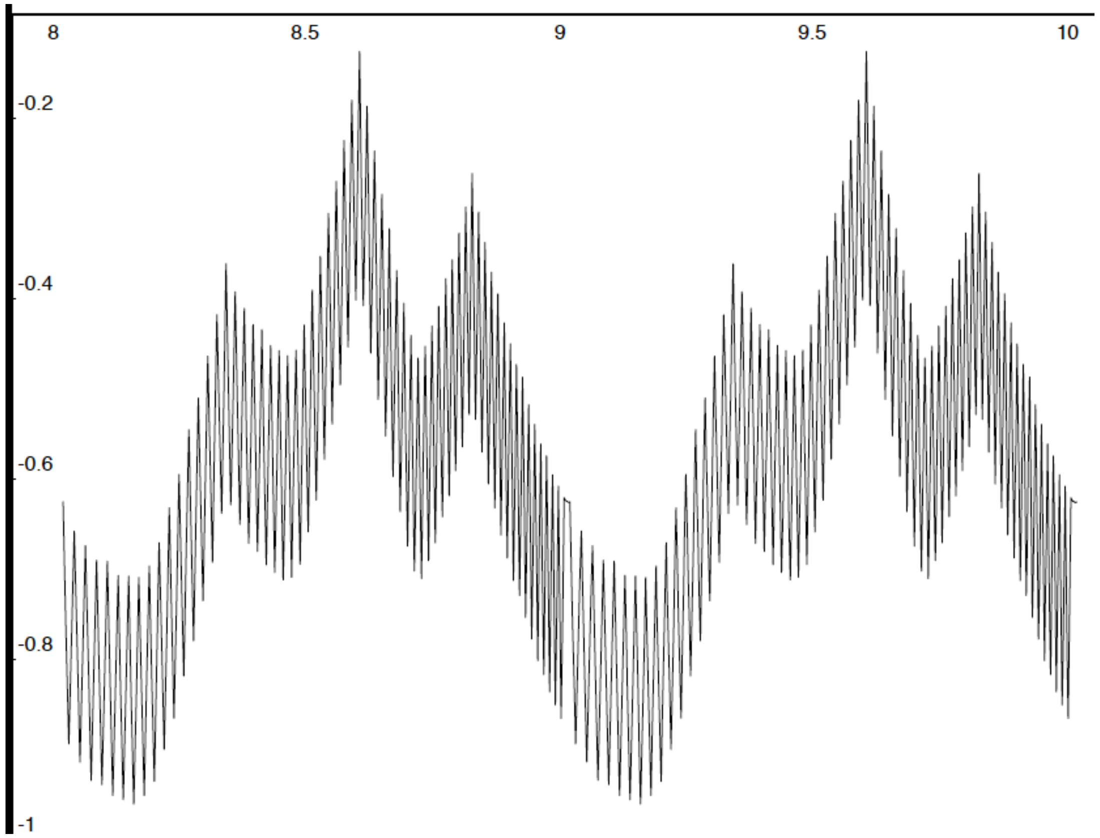
$$F(x) = \sum_k f_k e^{2\pi i k x}$$

where

$$f_0 = \frac{\lg \pi}{2} - \frac{1}{2\log 2} - \frac{1}{4}$$

$$f_k = \frac{1}{\log 2}\frac{\zeta(\chi_k)}{\chi_k(\chi_k+1)} \quad \text{for} \quad \chi_k = \frac{2\pi i k}{\log 2}$$

Example: Sum of Digits



$v_1(n)$ is # of "**1**"s in binary representation of n

$$f_n = \sum_{n \geq 1} v_1(n)$$

$$f_n = \frac{1}{2} n \lg n + nF(\lg n)$$

Diagram graphs $\frac{1}{n}\left(f_n - \frac{1}{2} n \lg n\right)$ for $2^8 \leq n \leq 2^{10}$

Can see continuity and periodicity of *F(u)*,
as well as fact that it is not continuously differentiable

Reprise

To calculate f_n

(A) Set $w_n = \Delta\nabla f_n$ and calculate Dirichlet Generating Function

$$W(s) = \sum_{n=1}^{\infty} \frac{w_n}{n^s} = \sum_{n=1}^{\infty} \frac{\Delta\nabla f_n}{n^s}$$

Observation: For D&C Recurrences and arithmetic functions this can be straightforward.

(B) Plug into Mellon-Perron formula and evaluate

$$f_n = n f_1 + \frac{n}{2i\pi} \int_{c-i\infty}^{c+i\infty} W(s) n^s \frac{ds}{s(s+1)}.$$

Observations: Done by showing this is equivalent to computing integral on contour and then calculating residues.
Equally spaced residues along a vertical line yield periodic functions.

Some Issues

General technique is very applicable but this presentation glossed over complications that sometimes arise.

- Proving convergence of Fourier series can be tricky. Sometimes don't have uniform convergence. (In those cases moving to triple summation occasionally works.)

- Only used special case of Mellin-Perron Formula that applied to double-summation. There are other versions that can be used for single summation, triple summation, etc.

- Showing that integral along other three sides of contour is negligible is not always easy.