Polynomial Time Algorithms for Constructing Optimal AIFV Codes

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Short Summary

Huffman encoding is an "optimal" lossless compression algorithm.

Optimality implicitly uses two unstated conditions:
(i) only one encoding (tree node) per source letter and
(ii) encoding is instantaneous.
    i.e., can decode a letter as soon as its final bit is seen.

Relaxing those two conditions permits constructing *Almost Instantaneous Fixed to Variable (AIFV)* code that beat Huffman.

Construction techniques are complicated:
using ellipsoid methods to find finite-state Markov Chains that have "optimal" steady state distributions.

Lots of open problems remaining.

Finding better AIFV codes.
Finding faster algorithms.
Finding strongly polynomial algorithms.

Outline

- Introduction
- AIFV-2 codes: cost and algorithm
- A Geometric Interpretation of the old algorithm
    - A New Binary Search Algorithm
    - An Ellipsoid Algorithm
- Extensions to AIFV-k codes (skip)
- Summing up and open questions

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- A *code* is a mapping C of source letters to codewords,
  e.g $C(a) = 01,\ C(b) = 0010,\ C(c) = 1001,\ C(d) = 001.$

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- $\Rightarrow$ the average code length is

$$L(C) = |C(a)|p_a + |C(b)|p_b + |C(c)|p_c + |C(d)|p_d$$

$$= 2 \times 0.5 + 3 \times 0.3 + 4 \times 0.15 + 4 \times 0.05 = 2.7$$

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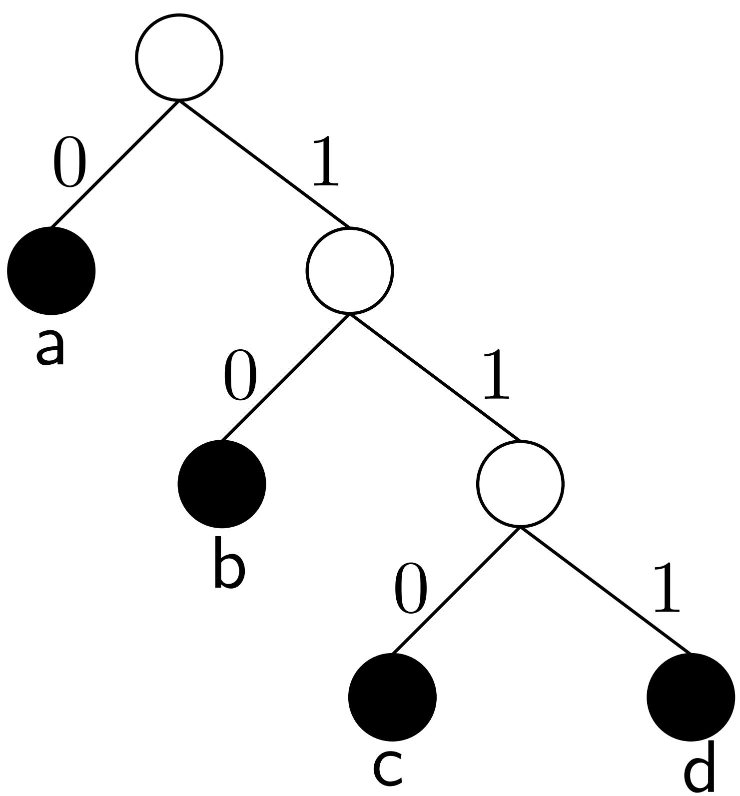

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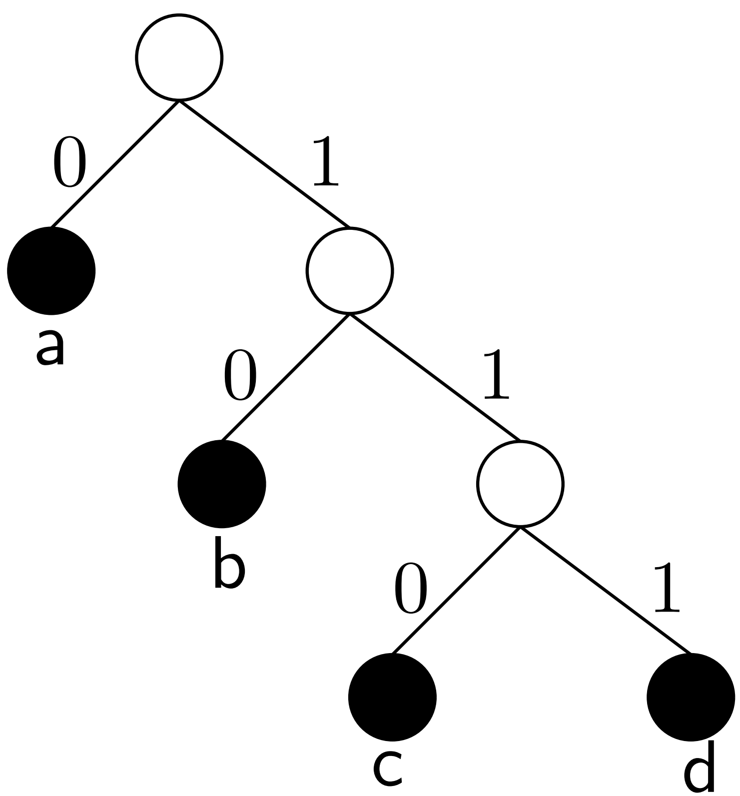


- Each leaf in tree corresponds to source letter $x \in \mathcal{X}$

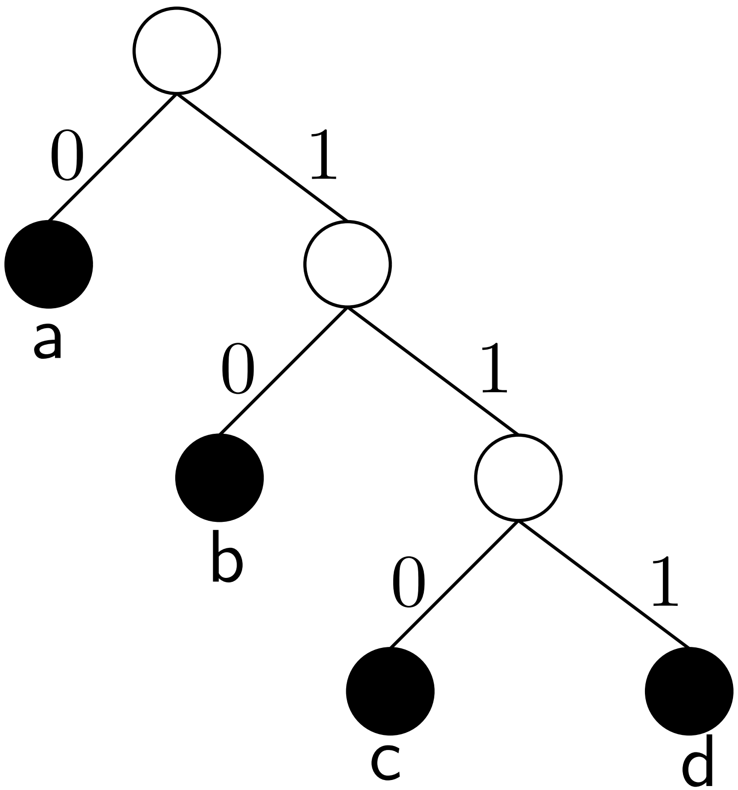
$$C(a) = 0$$

$$C(b) = 10$$

$$C(c) = 110$$

$$C(d) = 111$$

Given a Huffman Code, recall how to encode/decode.



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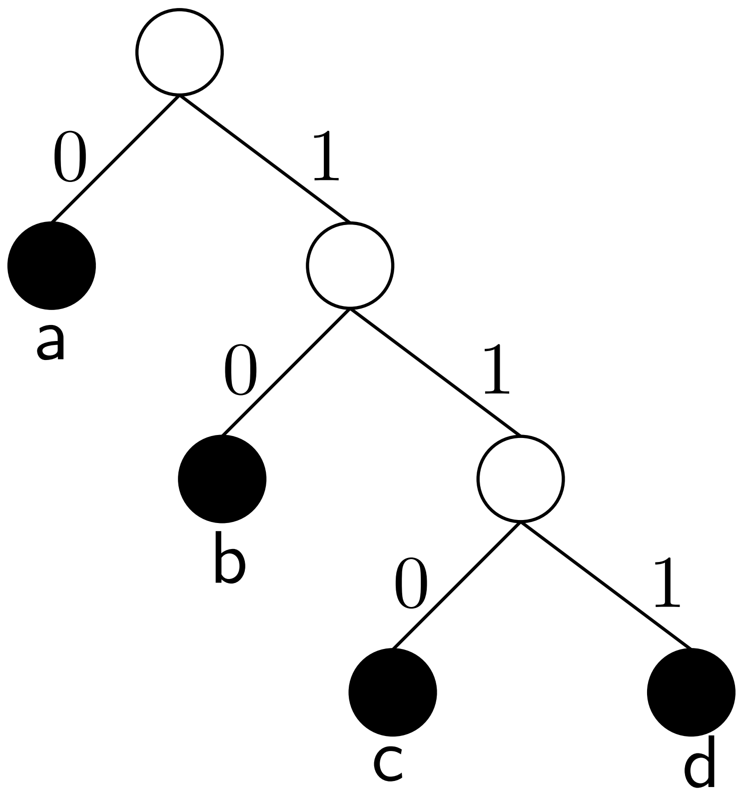

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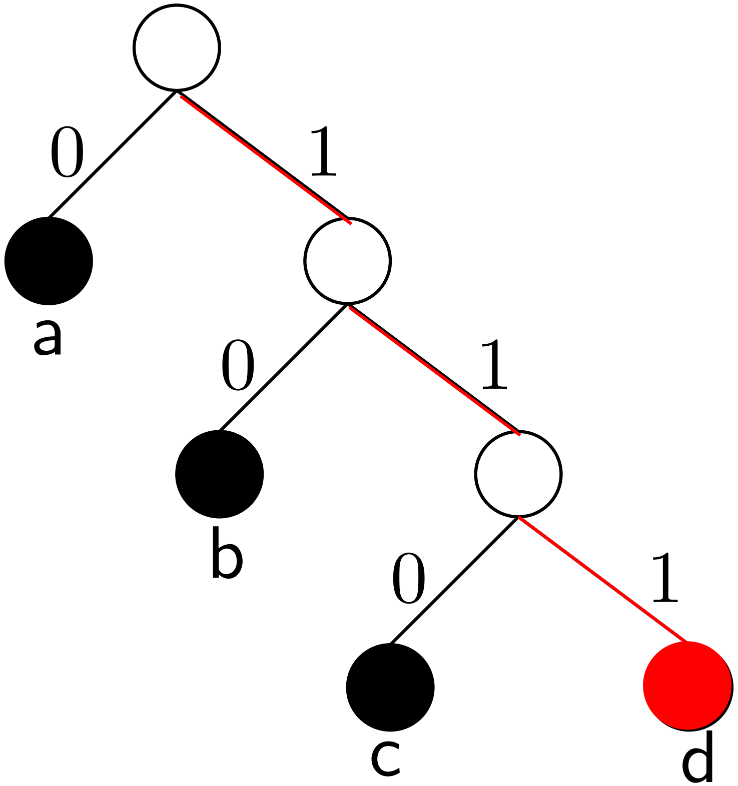

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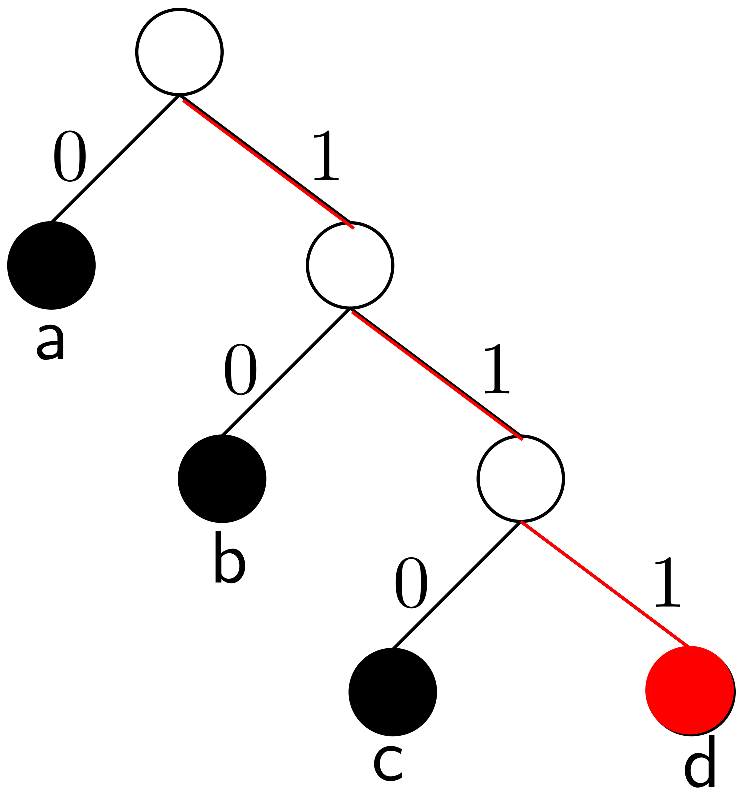

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$daba$ is encoded as 1110100

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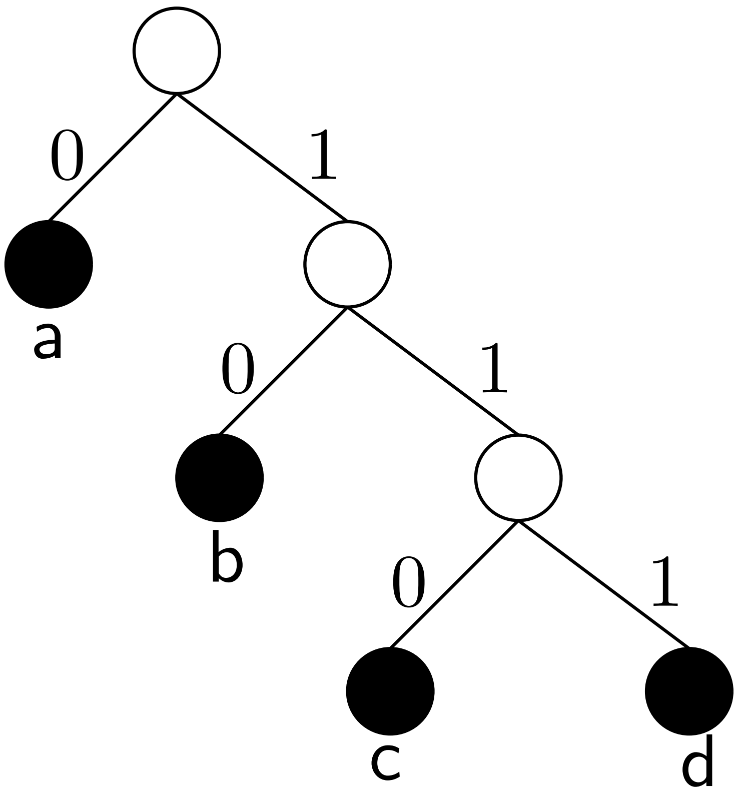

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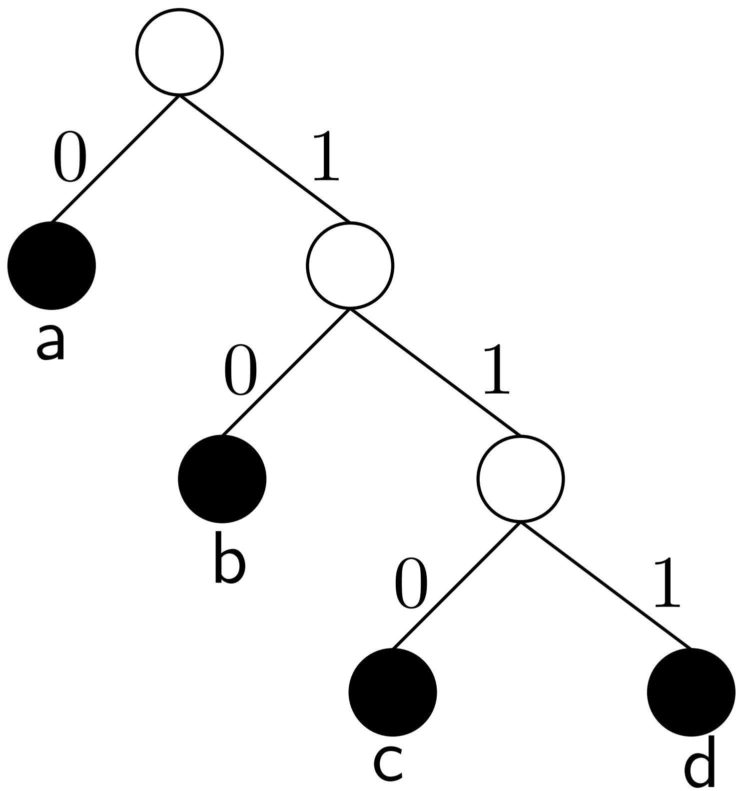

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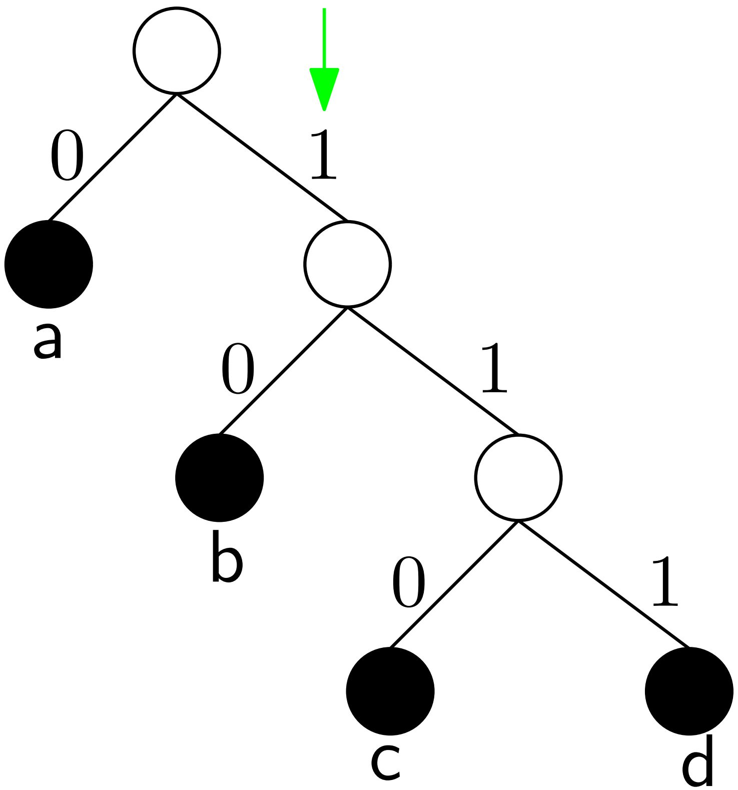

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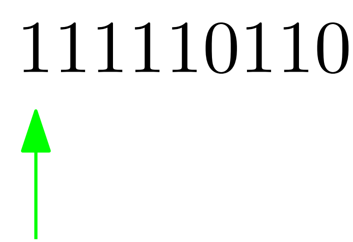

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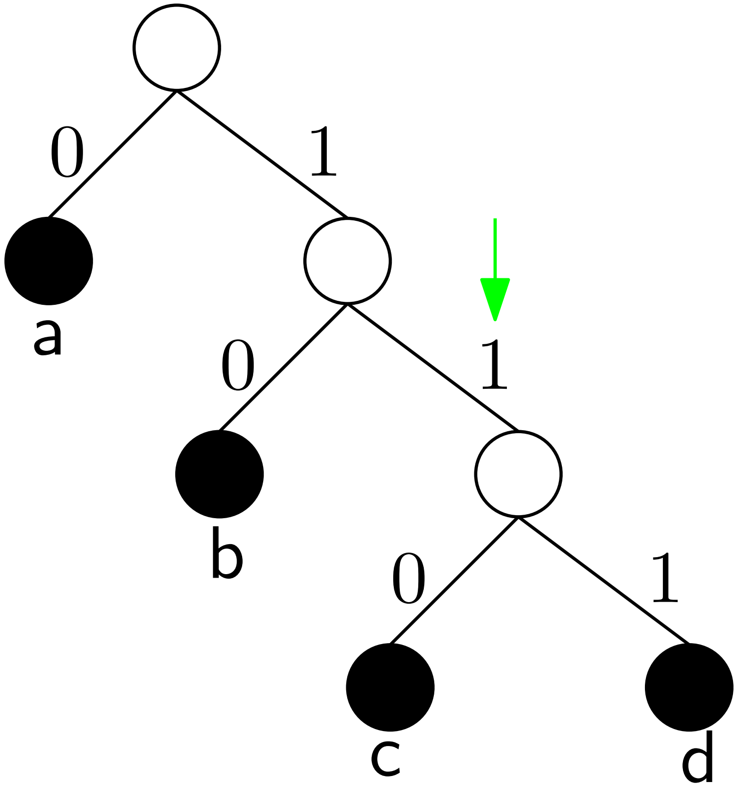

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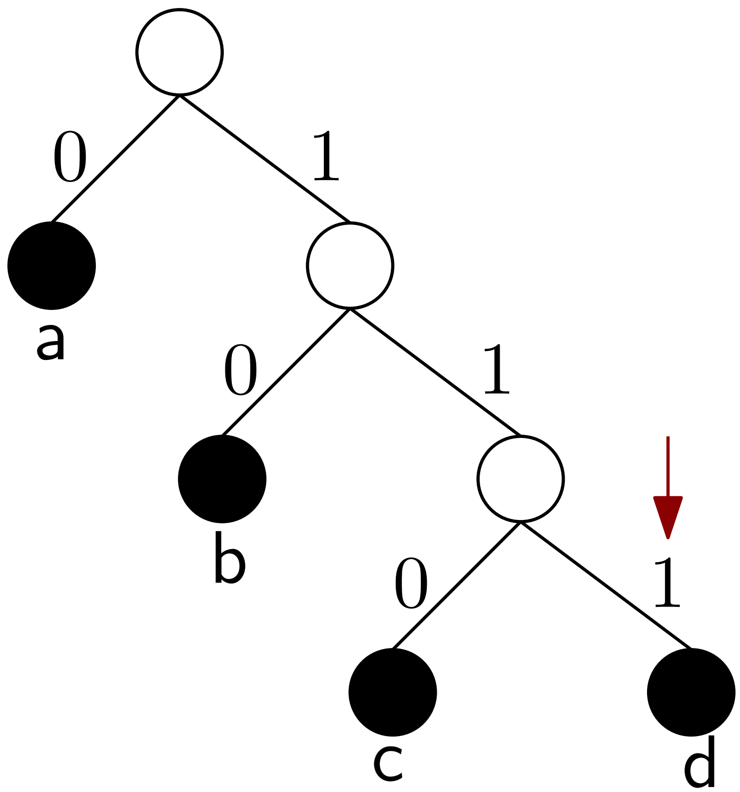

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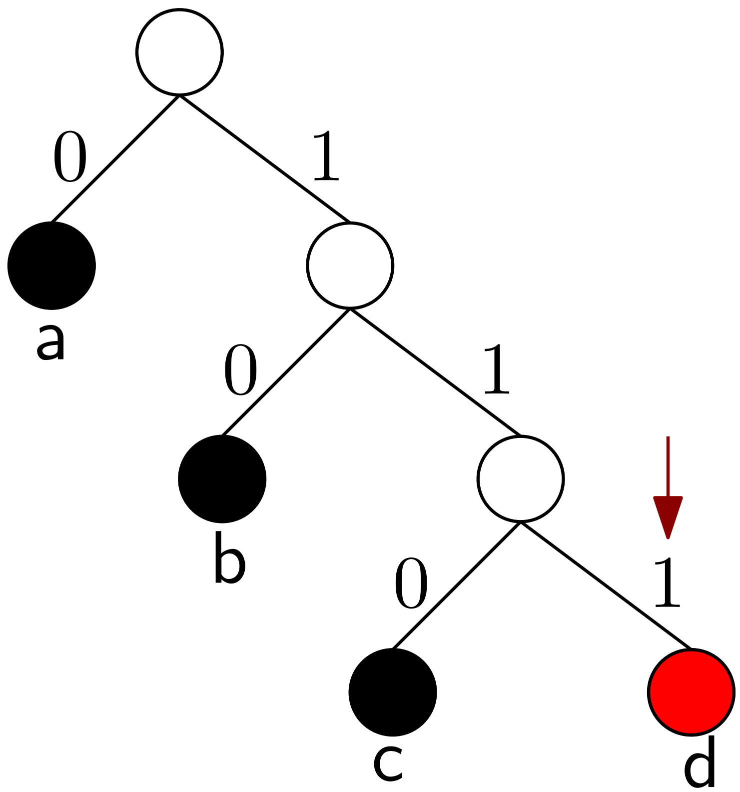

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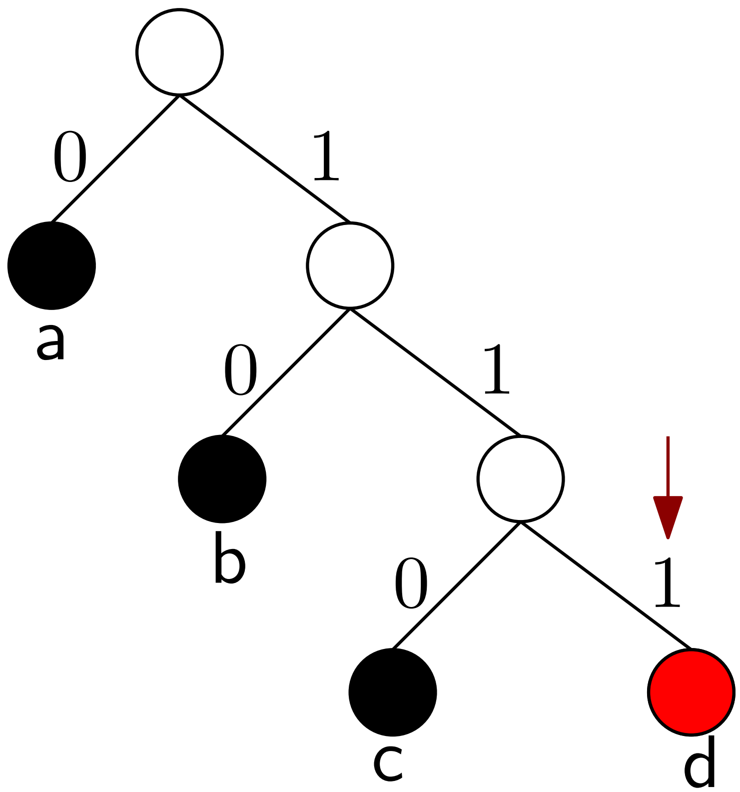

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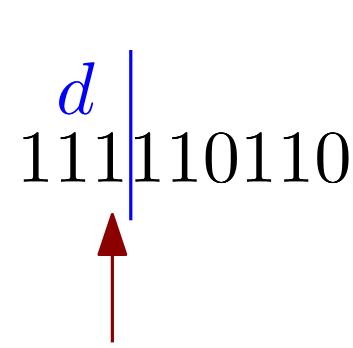

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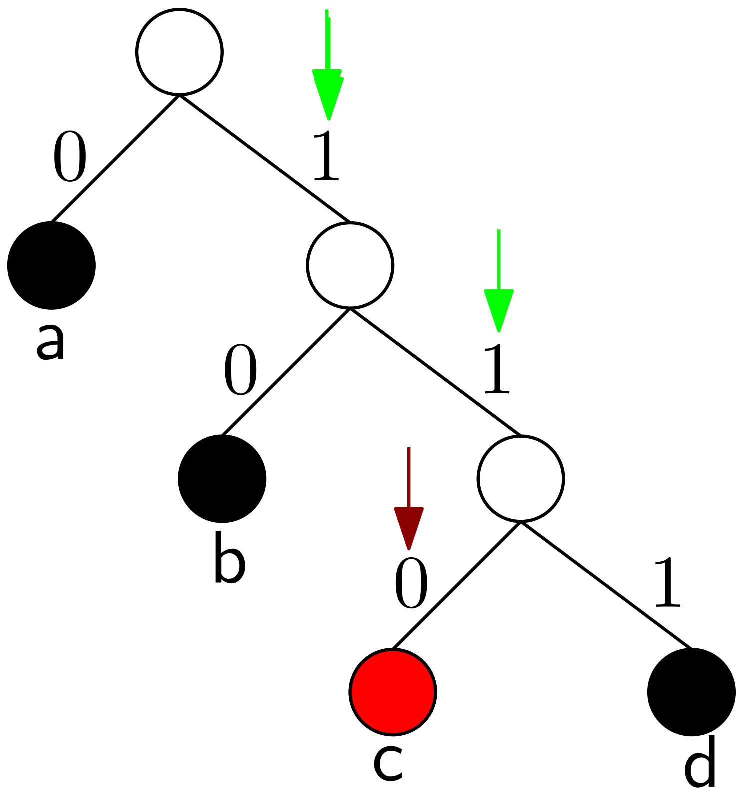

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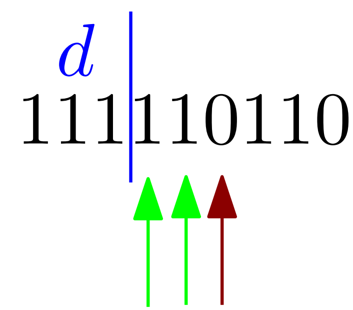

Stop! Reached leaf corresponding to c so decode as c.

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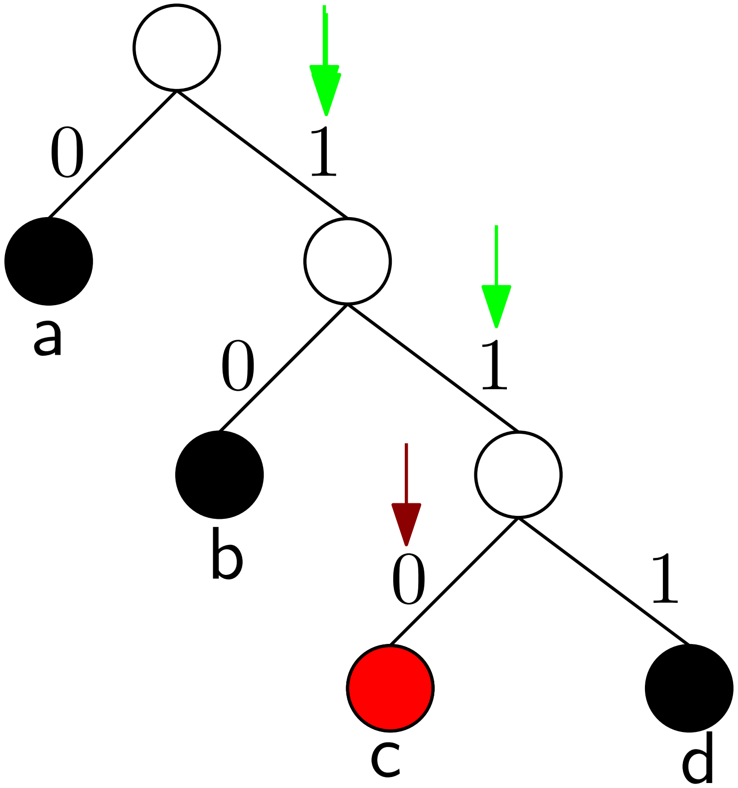

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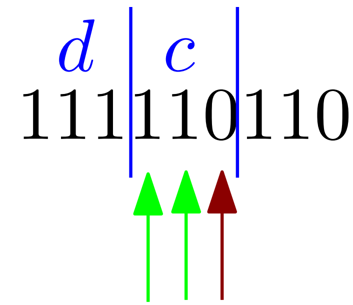

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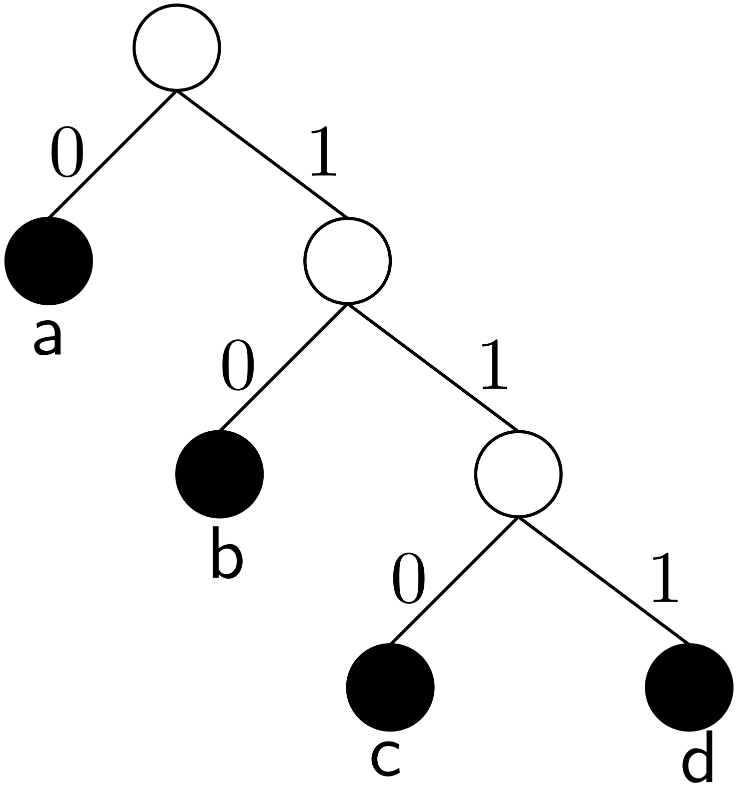

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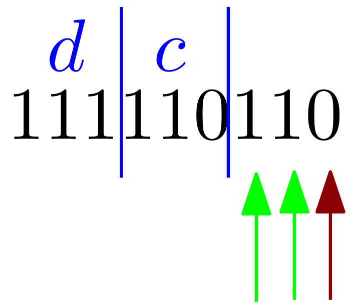

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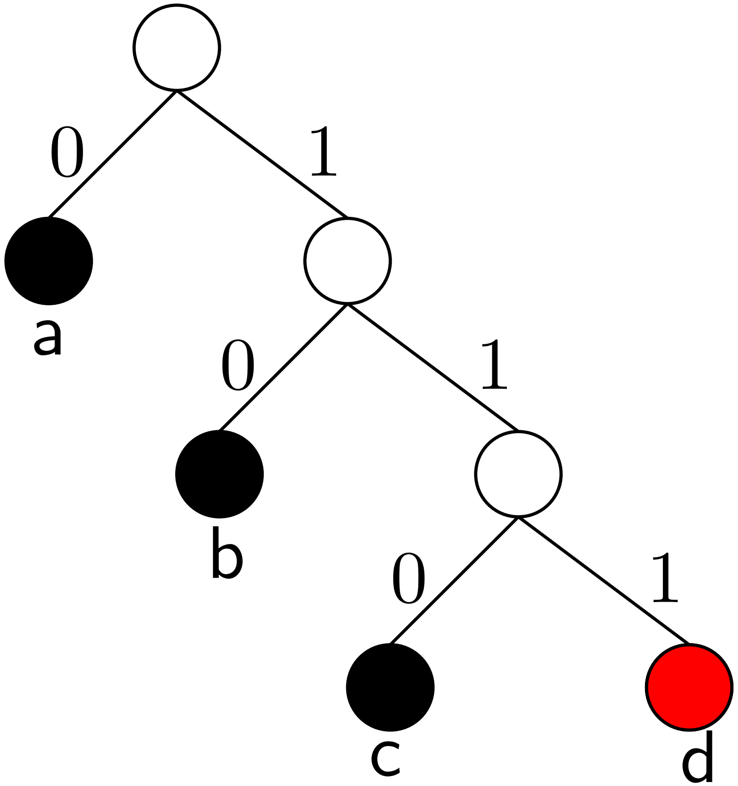

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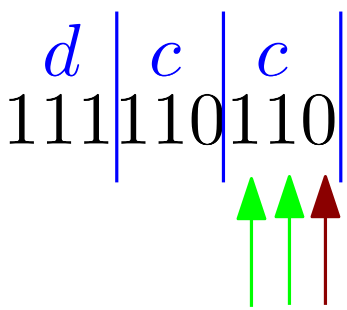

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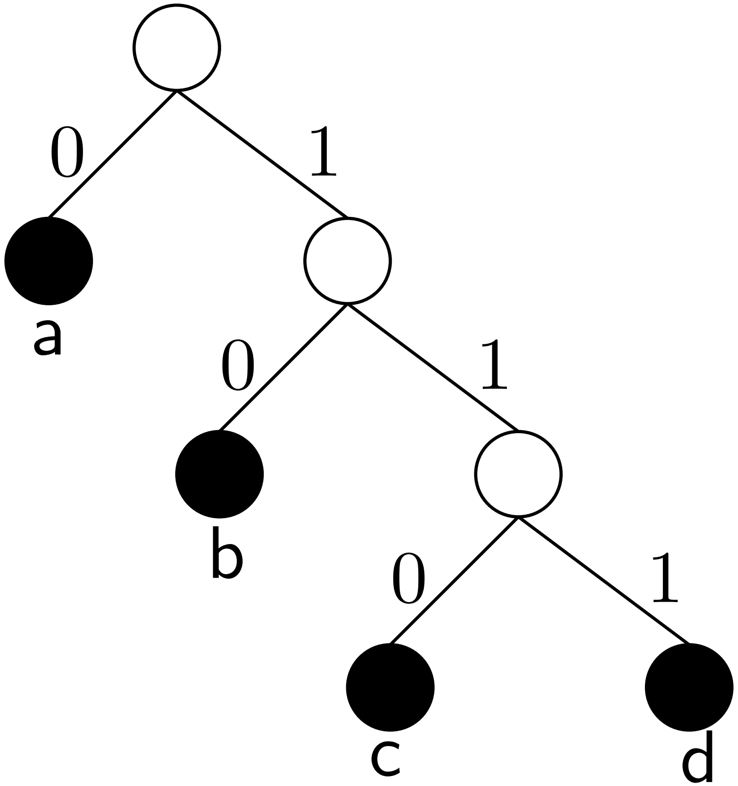

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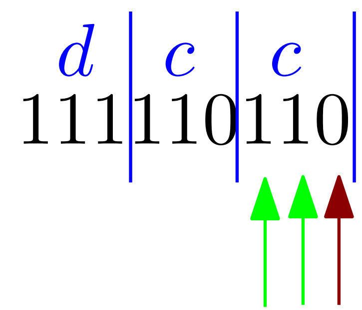

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Hence, 111110110 is decoded as dcc

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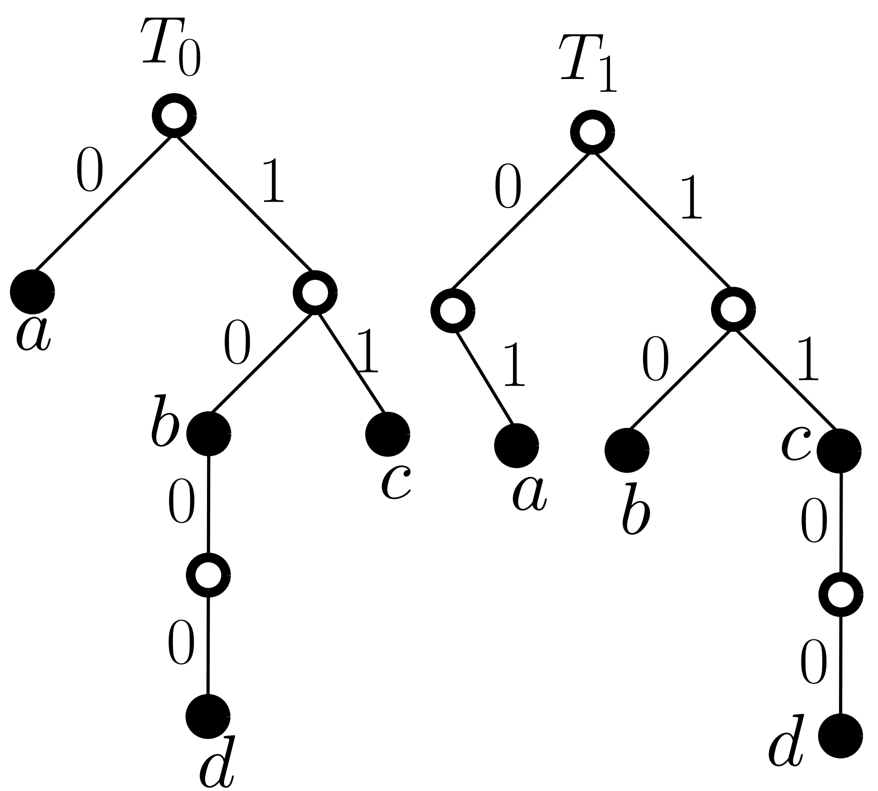

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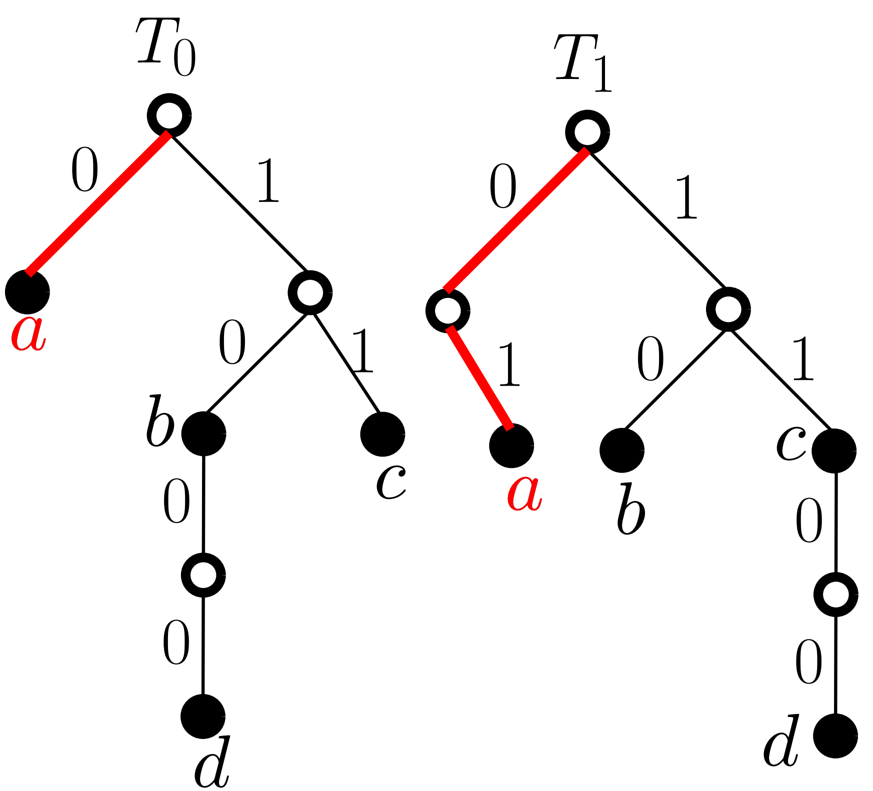

- $C_0(a) = 0,\ C_1(a) = 01$

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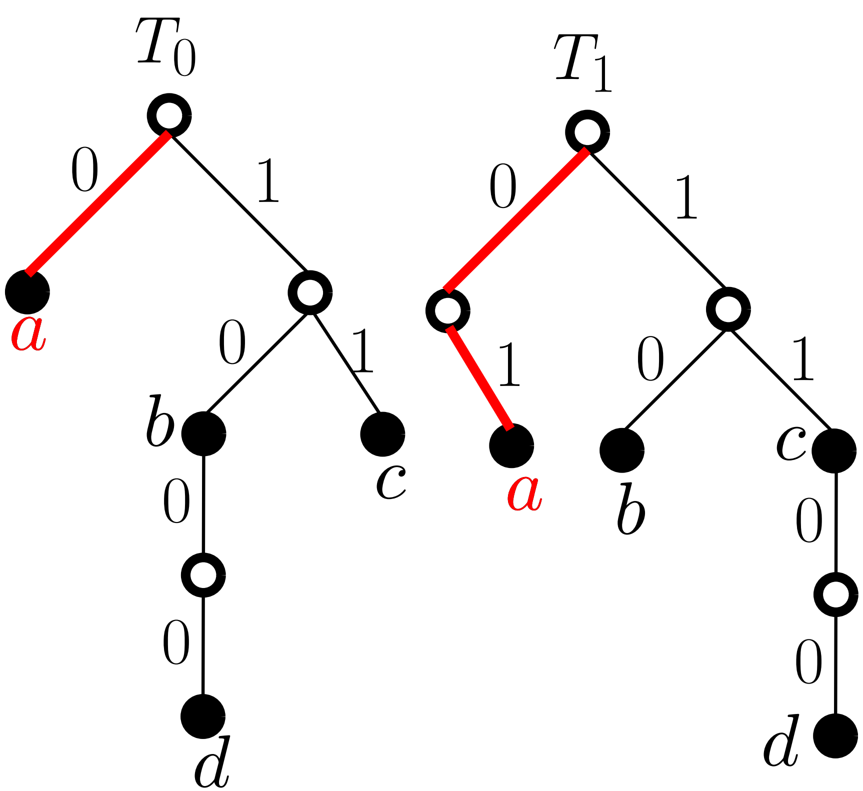

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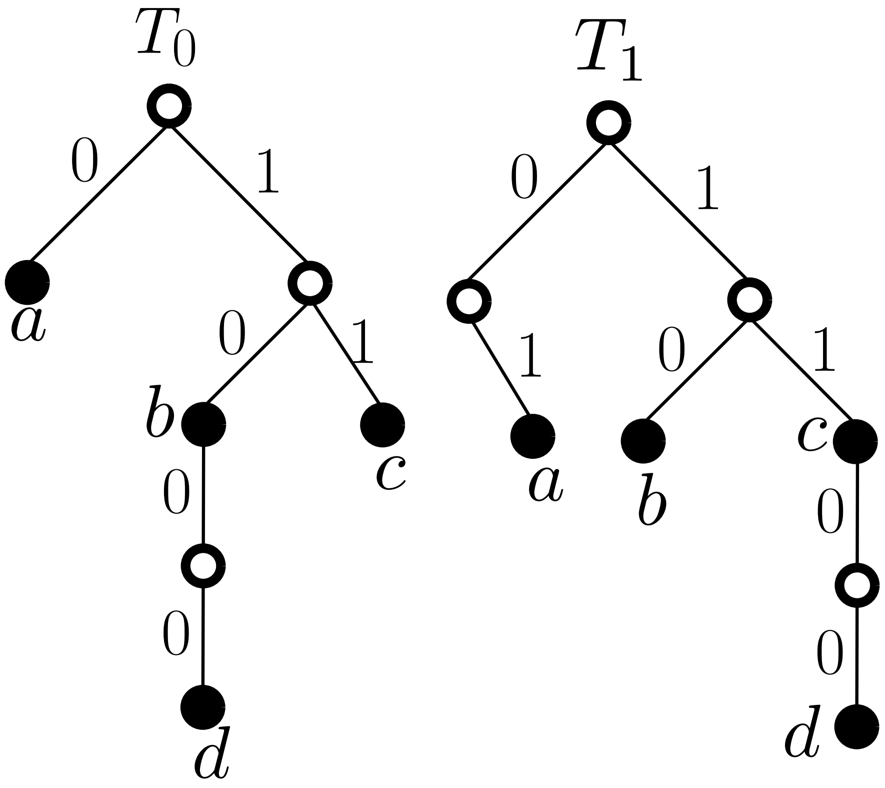
- $C_0(b) = 10$, $C_1(b) = 10$
- $C_0(c) = 11$, $C_1(c) = 11$
- $C_0(d) = 1000$, $C_1(d) = 1100$

Defintion of AIFV-2 Code T_0, T_1



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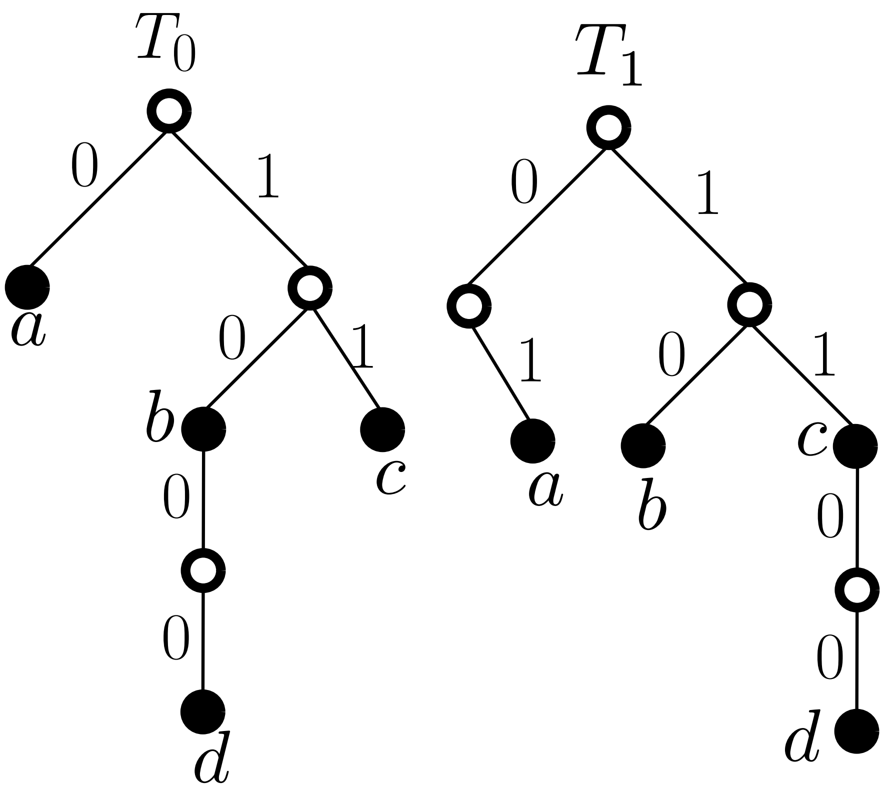

Root of T_1 is complete.

0 child of root only has a 1 child.

Incomplete internal nodes (with exception above) have only a 0 child.

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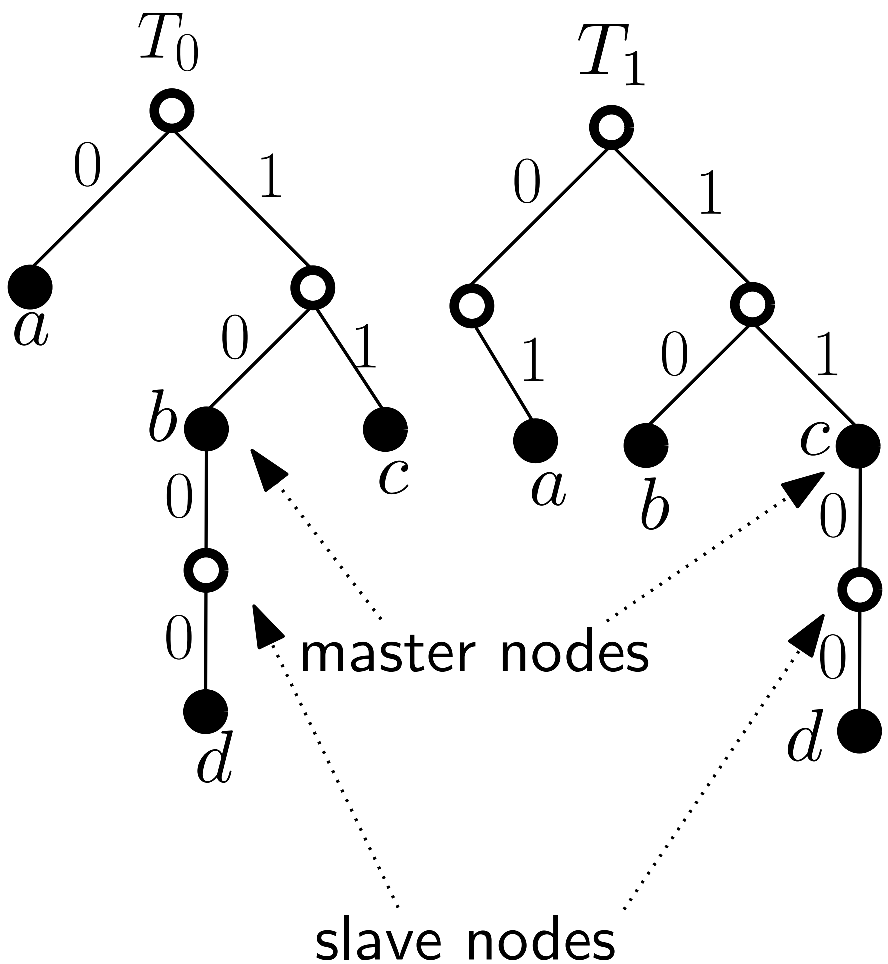

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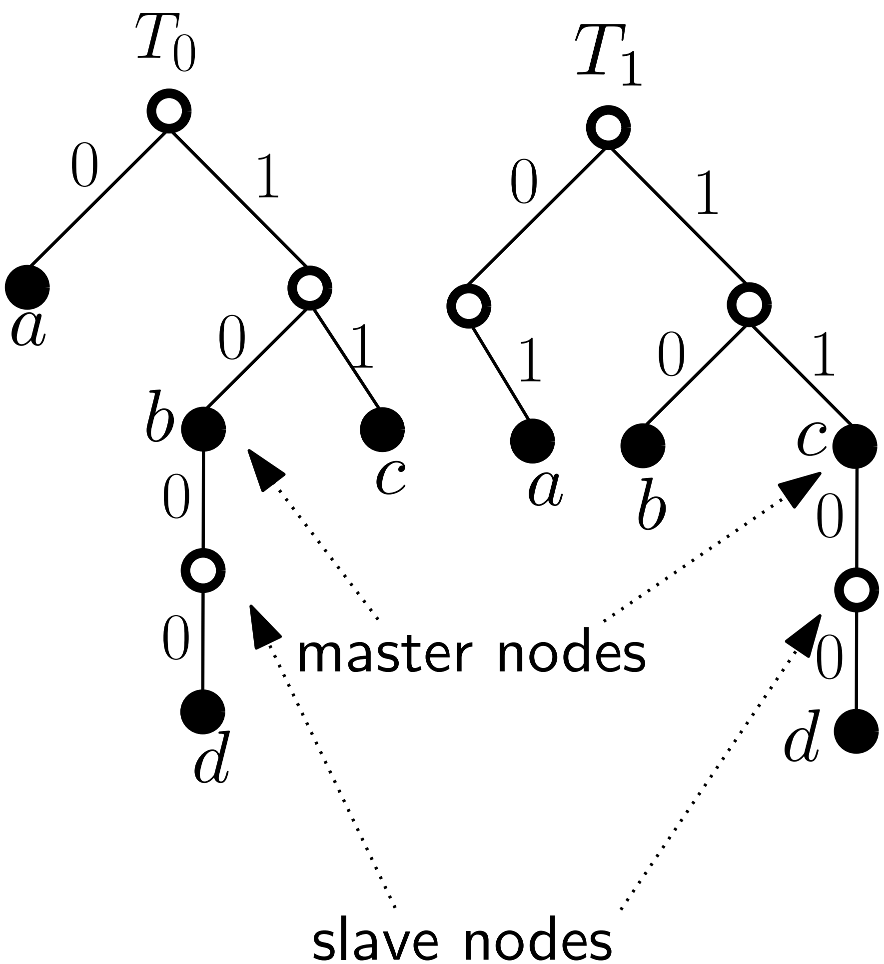

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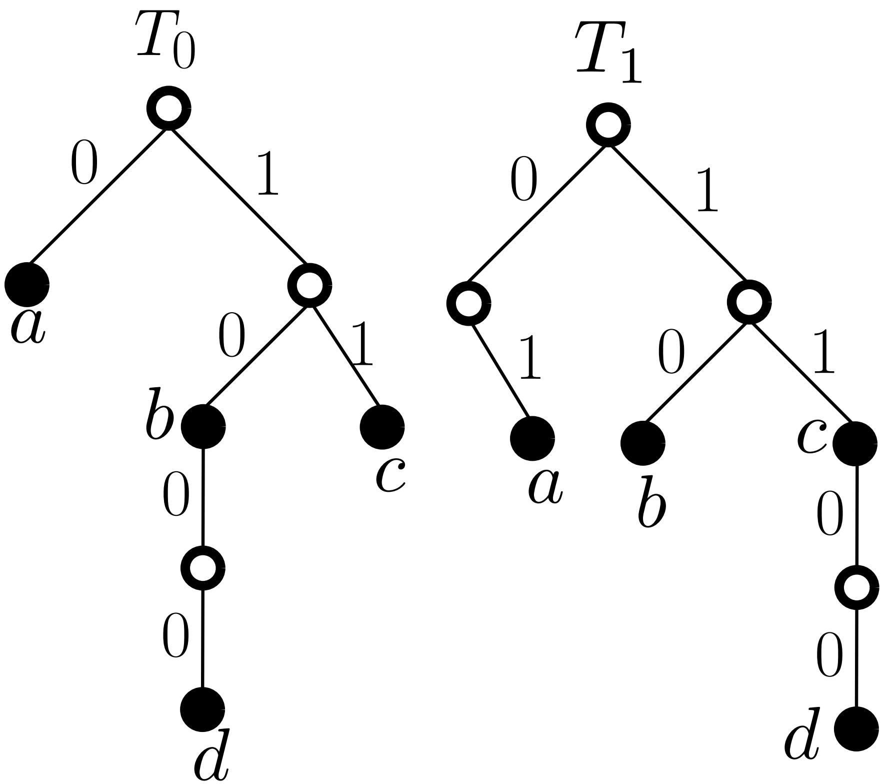
Codewords are leaves and master nodes.
Slave nodes and complete internal nodes are **not** codewords.

Encoding/Decoding with AIFV-2 Codes T_0, T_1

Encoding $S = s_1, s_2, \ldots s_k \in \mathcal{X}^k$



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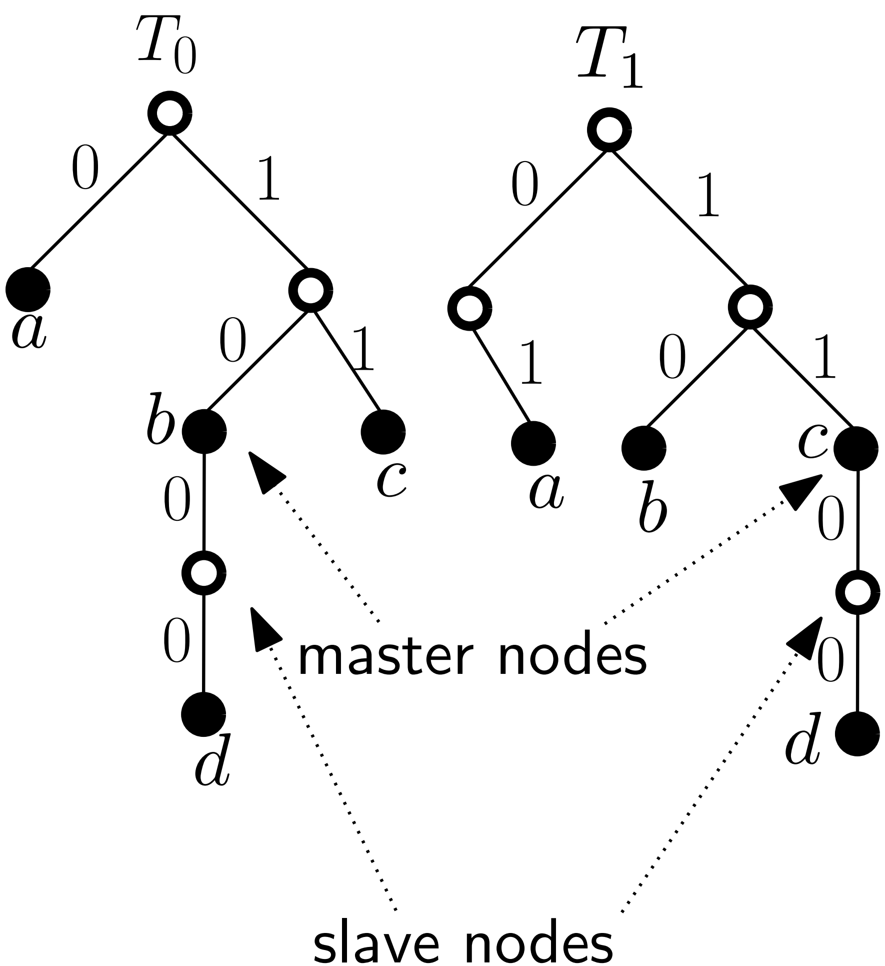

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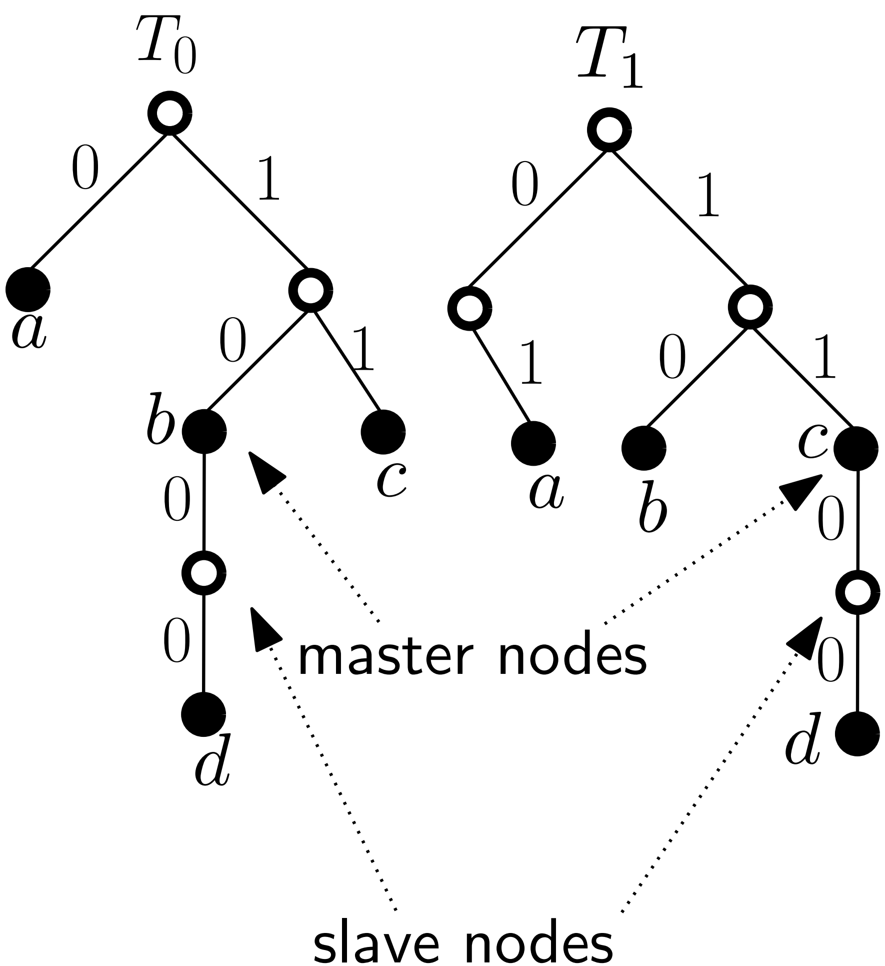
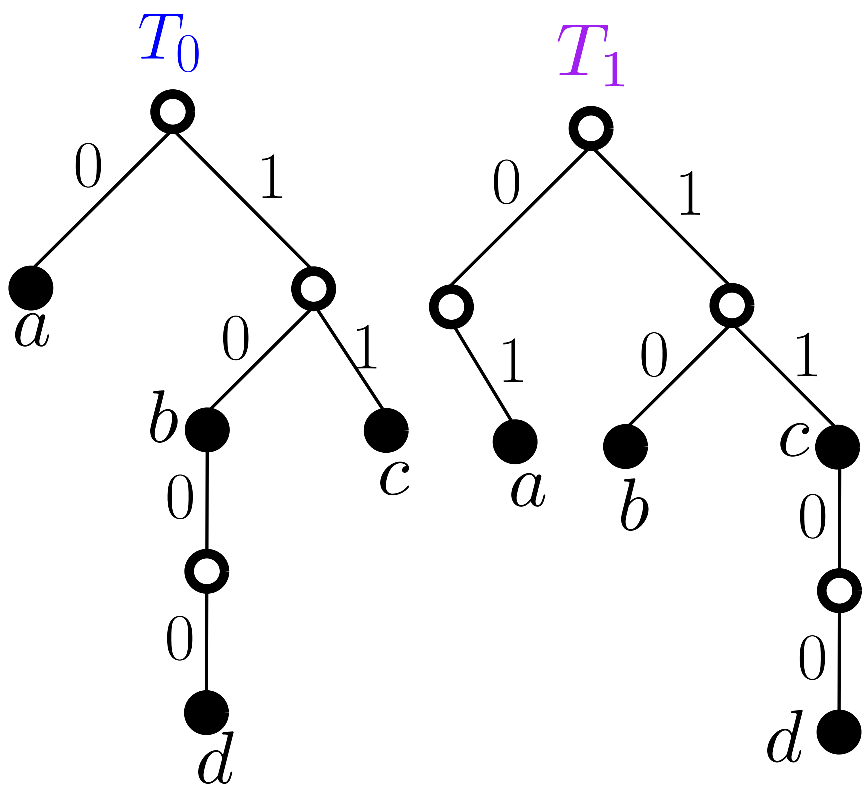


Encode s_1 with tree T_0

For $i = 2$ to k
  if s_{i-1} was encoded
    using a master node
      encode s_i with tree T_1
    else:
      encode s_i with tree T_0

Example: Encoding $dabcab$



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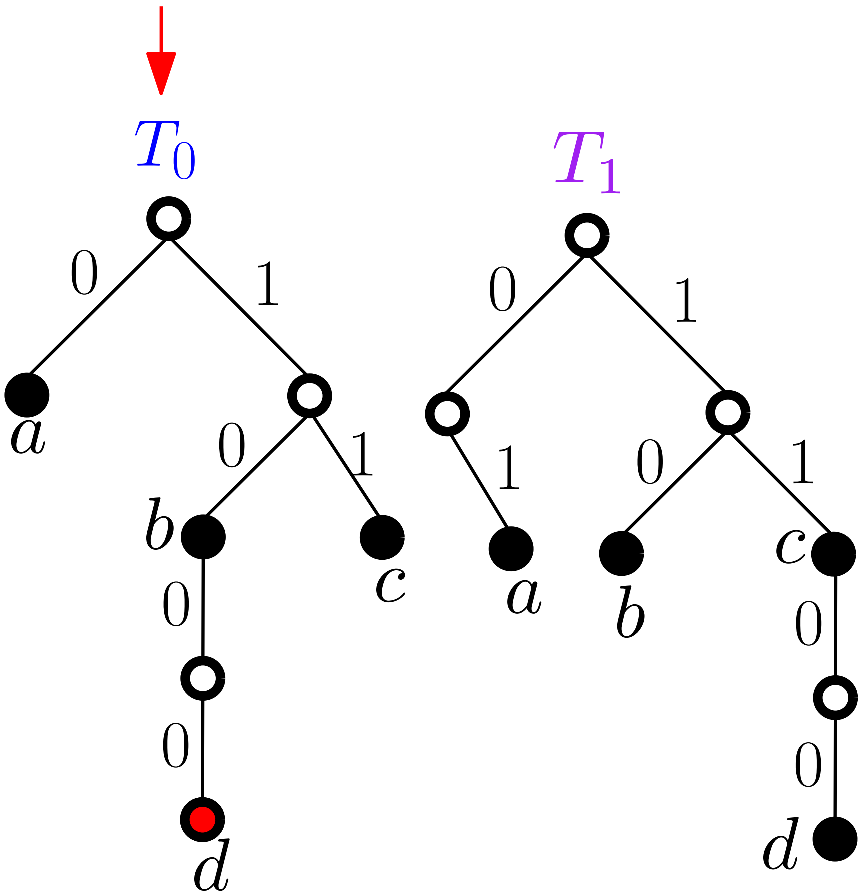

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Start in T_0.

Encode d as $C_0(d) = 1000$

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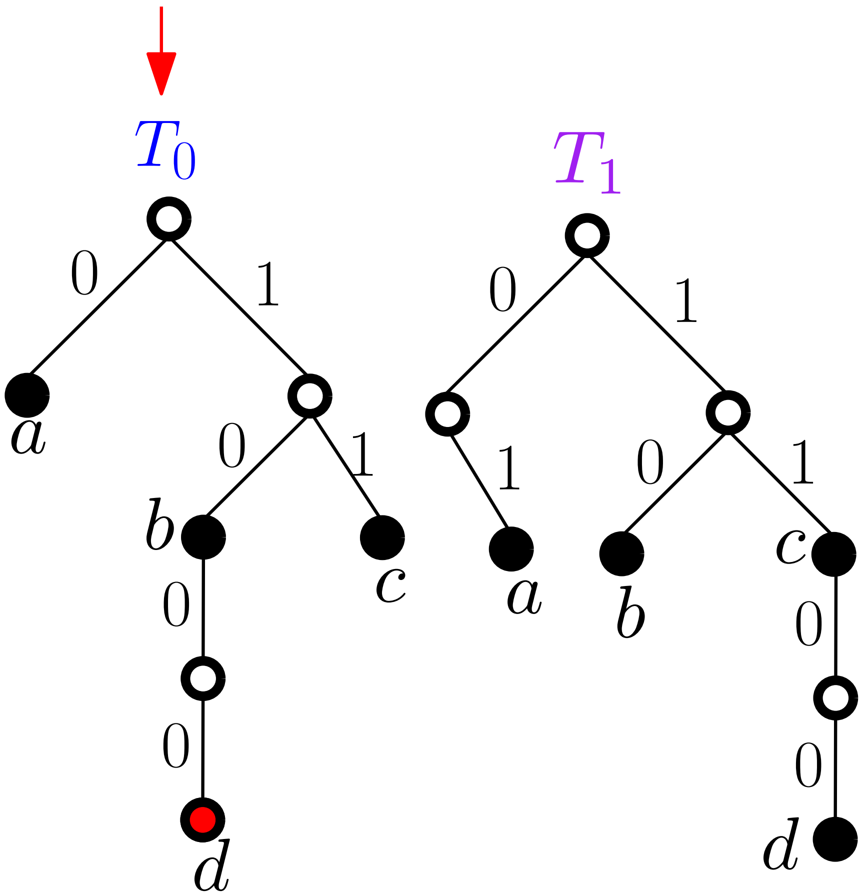

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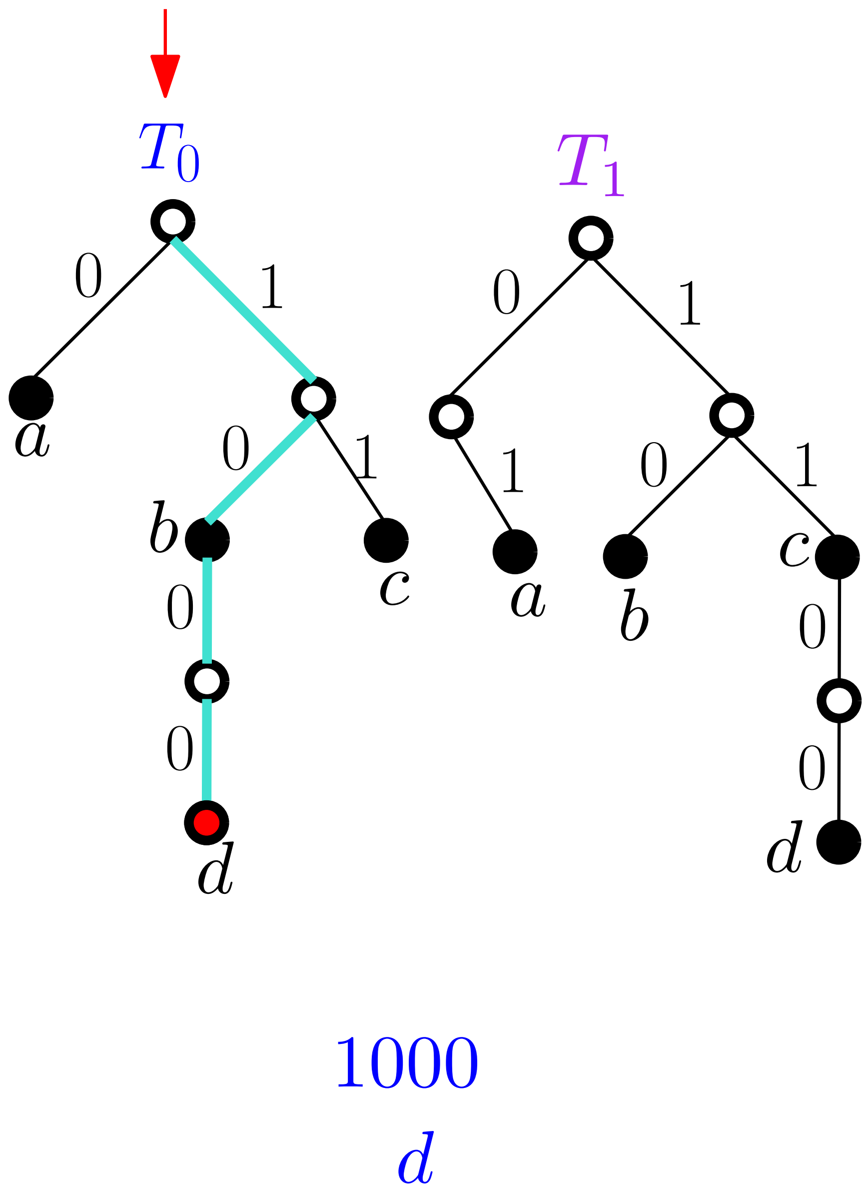

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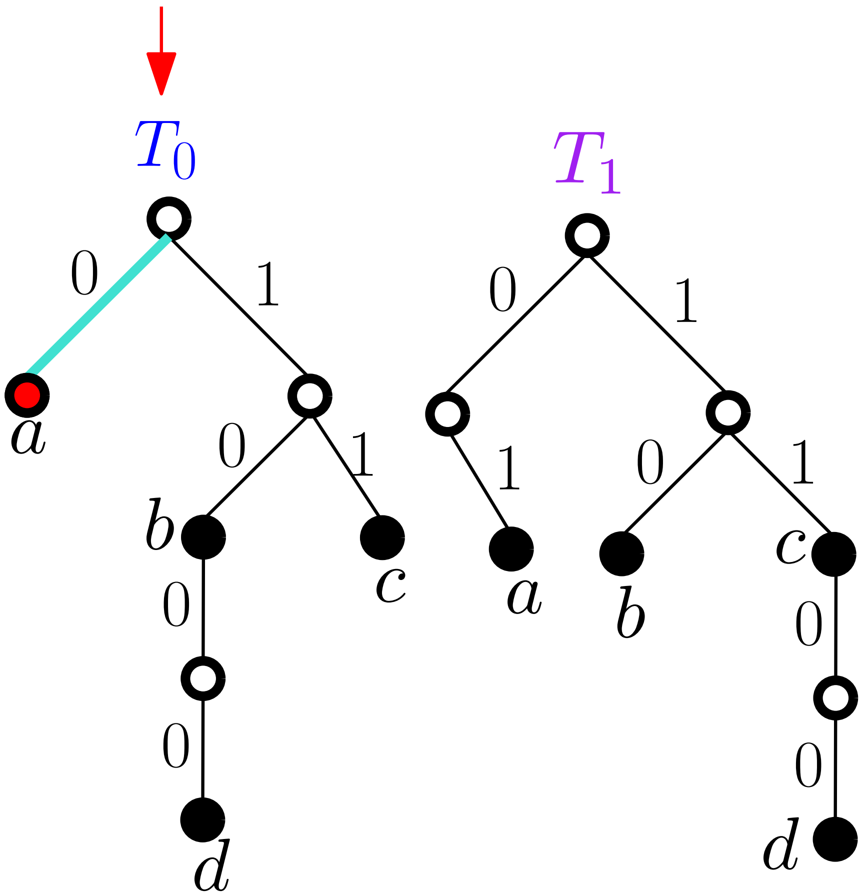
d is not master $\Rightarrow$ stay in T_0

Example: Encoding $dabcab$



$dabcab$

Start in T_0.

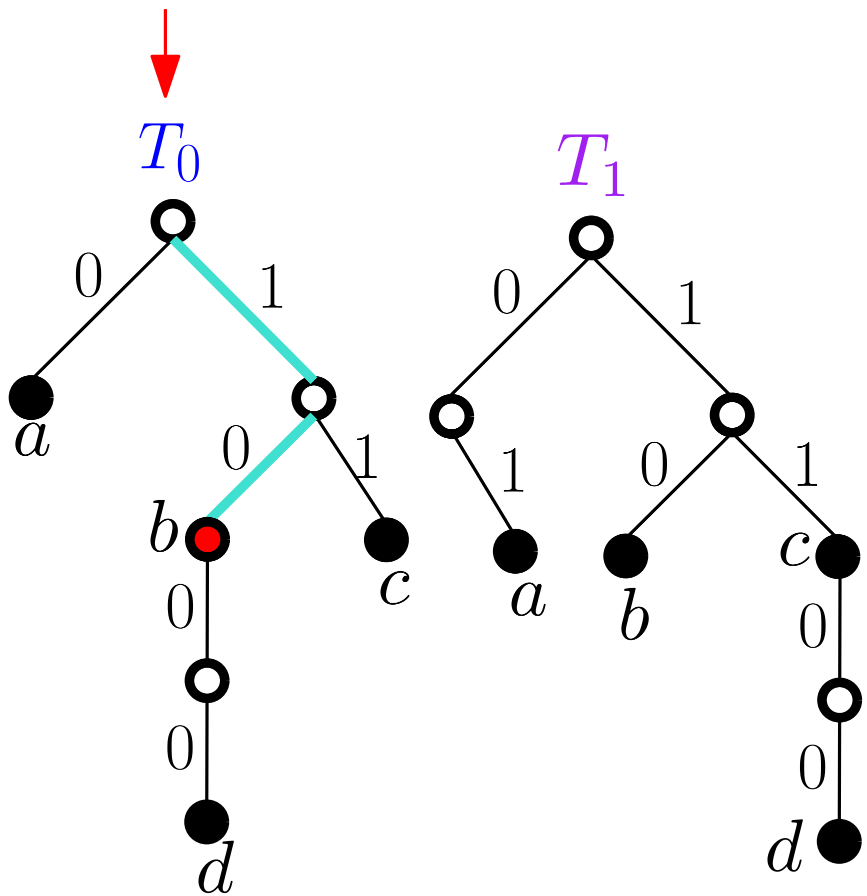
Encode a as $C_0(a) = 0$

a is not master $\Rightarrow$ stay in T_0

$1000\ 0$

$d\quad a$

Example: Encoding $dabcab$



$dabcab$

Start in T_0.

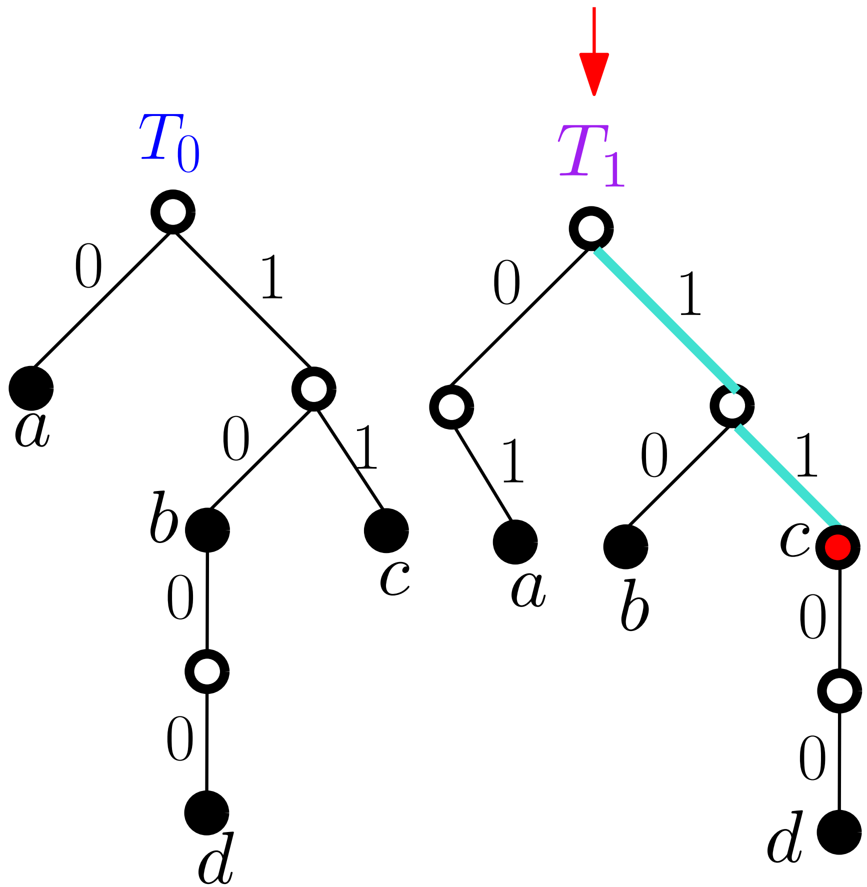
Encode b as $C_0(b) = 10$

b is a master $\Rightarrow$ switch to T_1

$1000 \; 0 \; 10$

$d \quad a \quad b$

Example: Encoding $dabcab$



$dabcab$

Start in T_1.

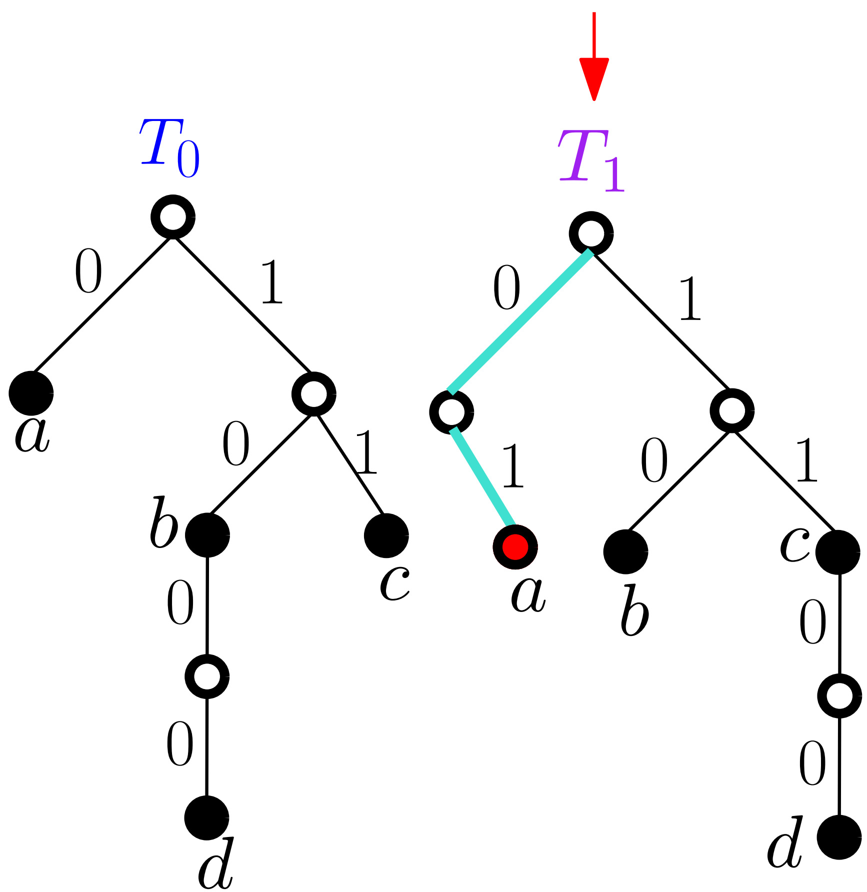
Encode c as $C_1(c) = 11$

c is a master $\Rightarrow$ stay in T_1

$1000\ 0\ \ 10\ 11$

$d\ \ a\ \ b\ \ c$

Example: Encoding *dabcab*



$dabcab$

Start in T_1.

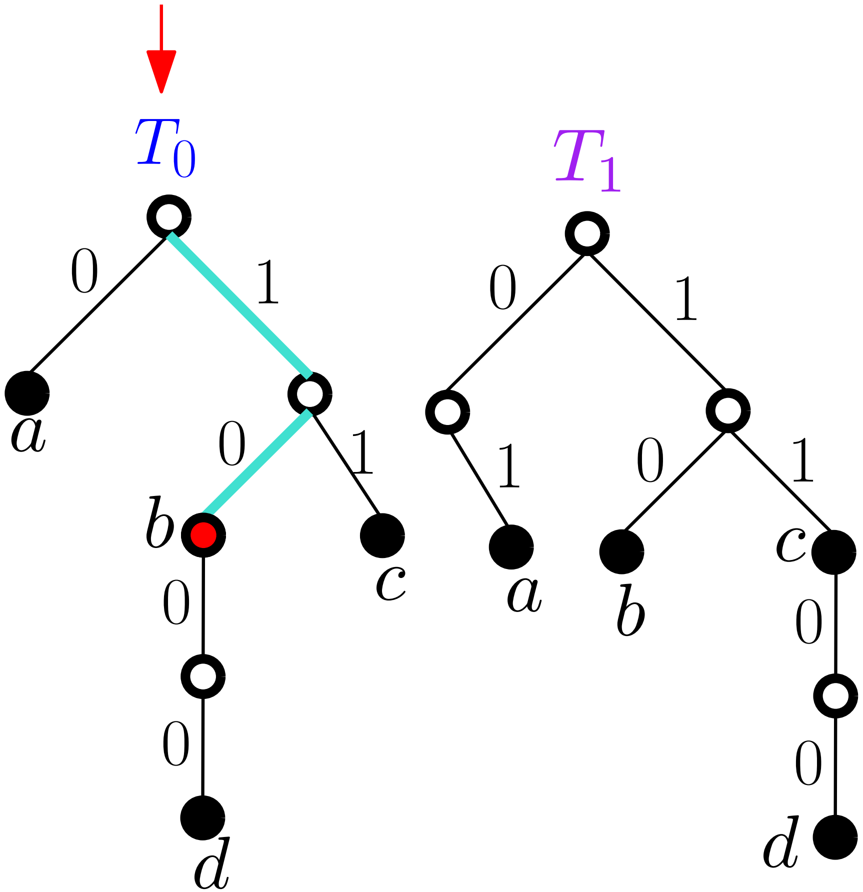
Encode a as $C_1(a) = 01$

a is not a master $\Rightarrow$ switch to T_0

$1000\ 0\ \ 10\ 11\ 01$

$d \quad a \quad b \quad c \quad a$

Example: Encoding *dabcab*



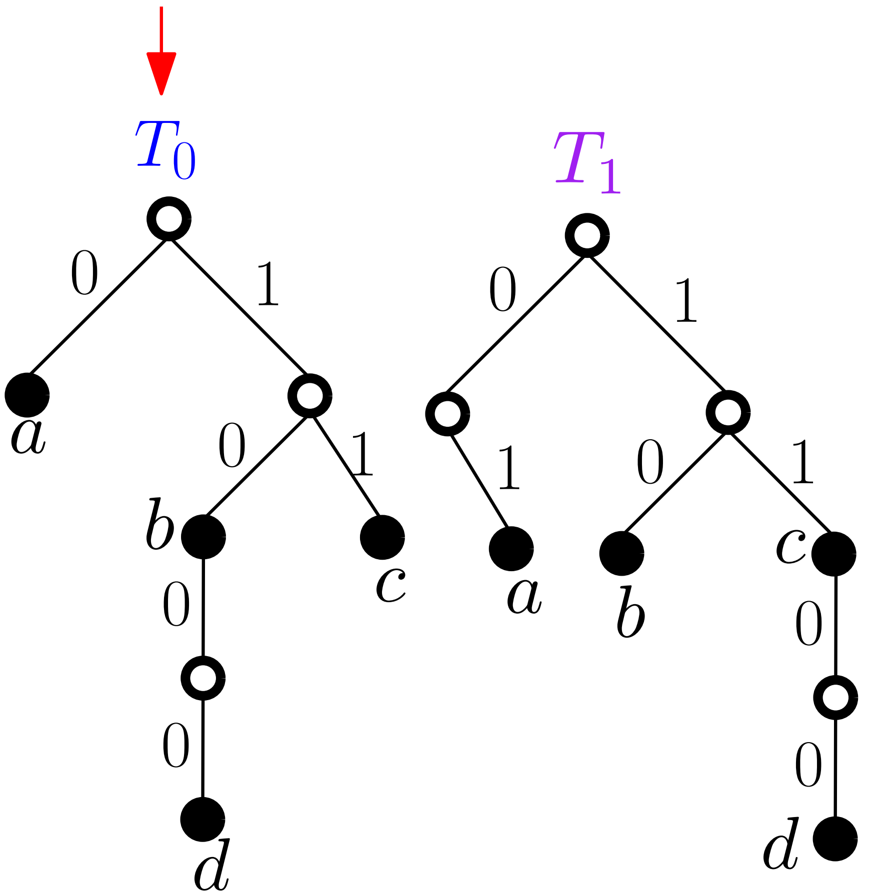
dabcab

Start in T_0.
Encode b as $C_0(b) = 10$

$1000\ 0\ \ 10\ 11\ 01\ 10$

$d\quad a\quad b\quad c\quad a\quad b$

Example: Encoding $dabcab$

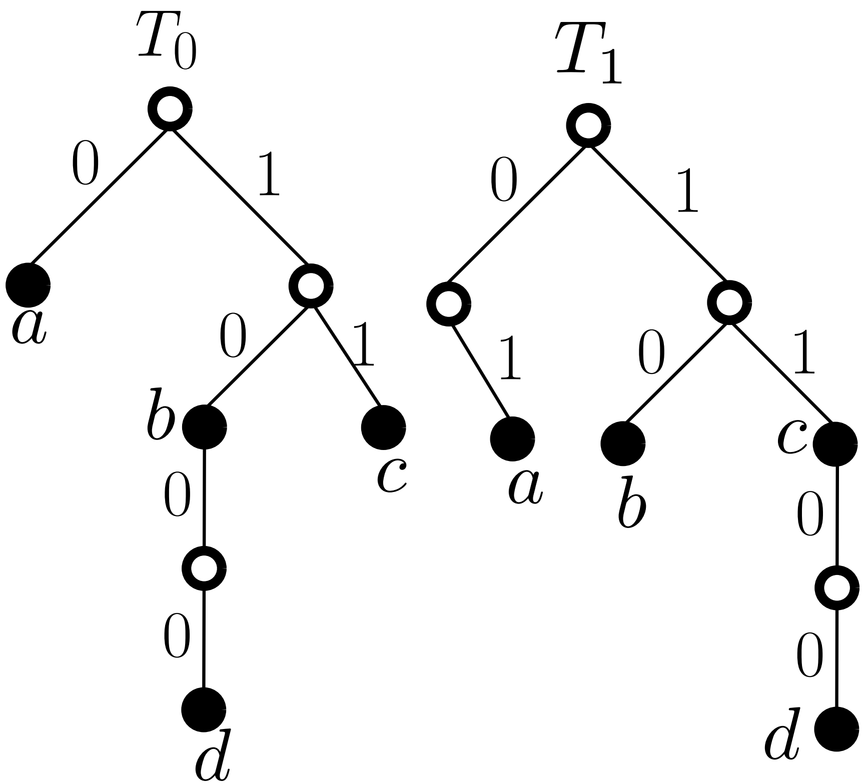


$1000\ 0\ \ 10\ 11\ 01\ 10$ ← Encoding of $dabcab$

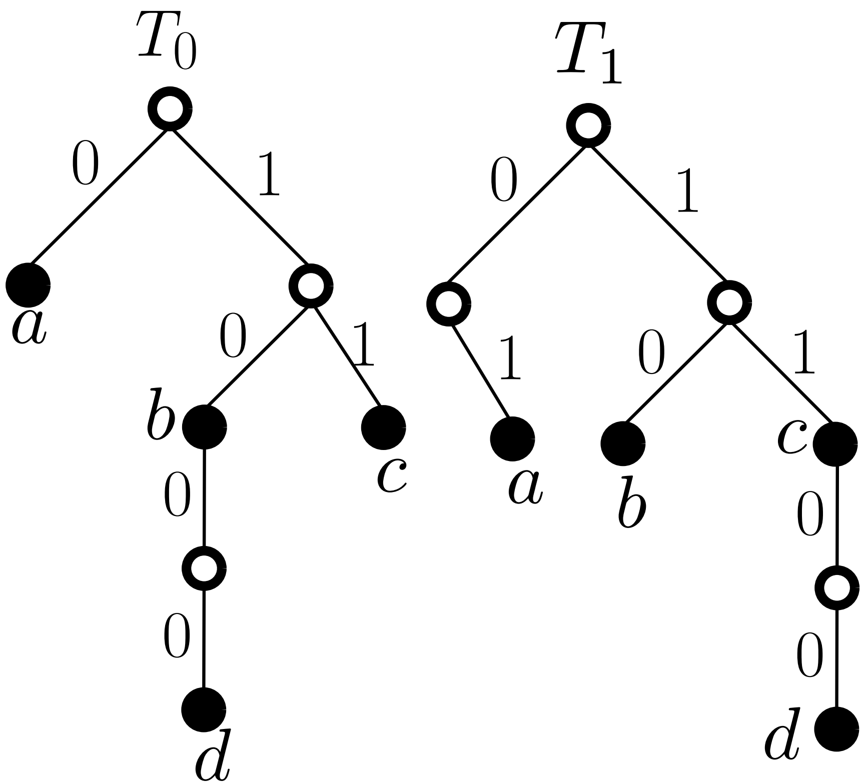
$d\quad a\quad b\quad c\quad a\quad b$

The Decoding Procedure

Start at T_0 and trace codeword through tree.



The Decoding Procedure

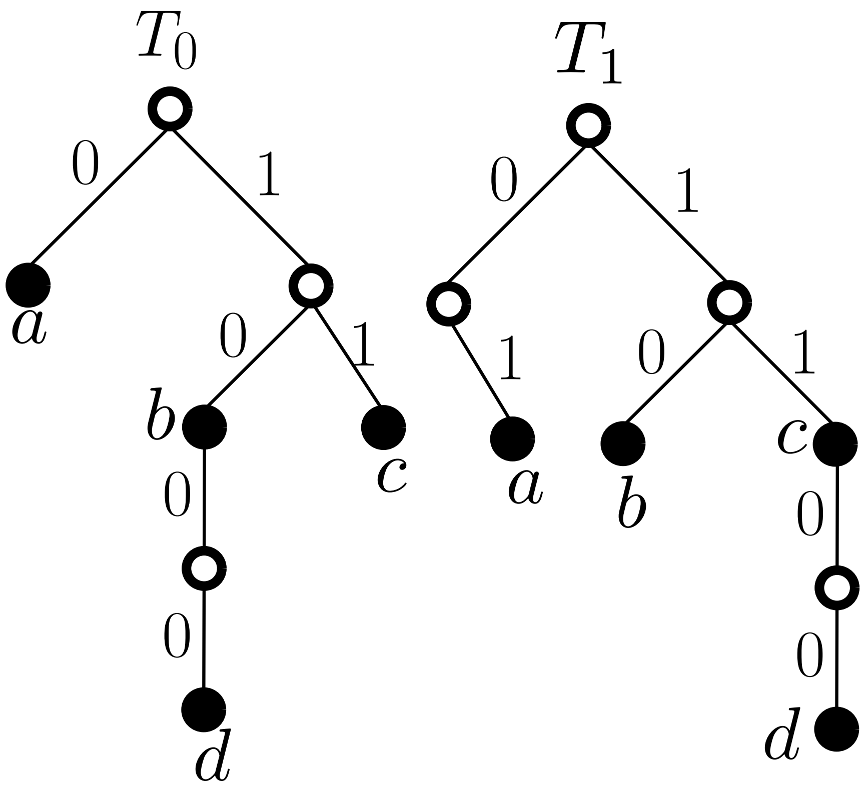


Start at T_0 and trace codeword through tree.

If a leaf is reached, decode using that word.

If decoding is "blocked" due to missing "1" edge, go back to last master seen and use it as decoded letter.

The Decoding Procedure



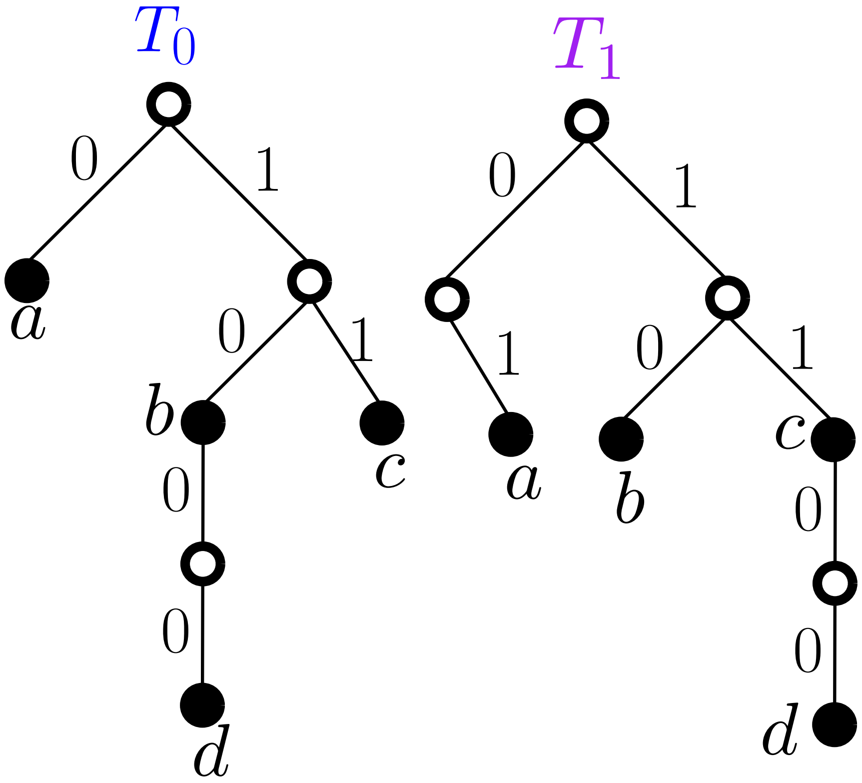
Start at T_0 and trace codeword through tree.

If a leaf is reached, decode using that word.

If decoding is "blocked" due to missing "1" edge, go back to last master seen and use it as decoded letter.

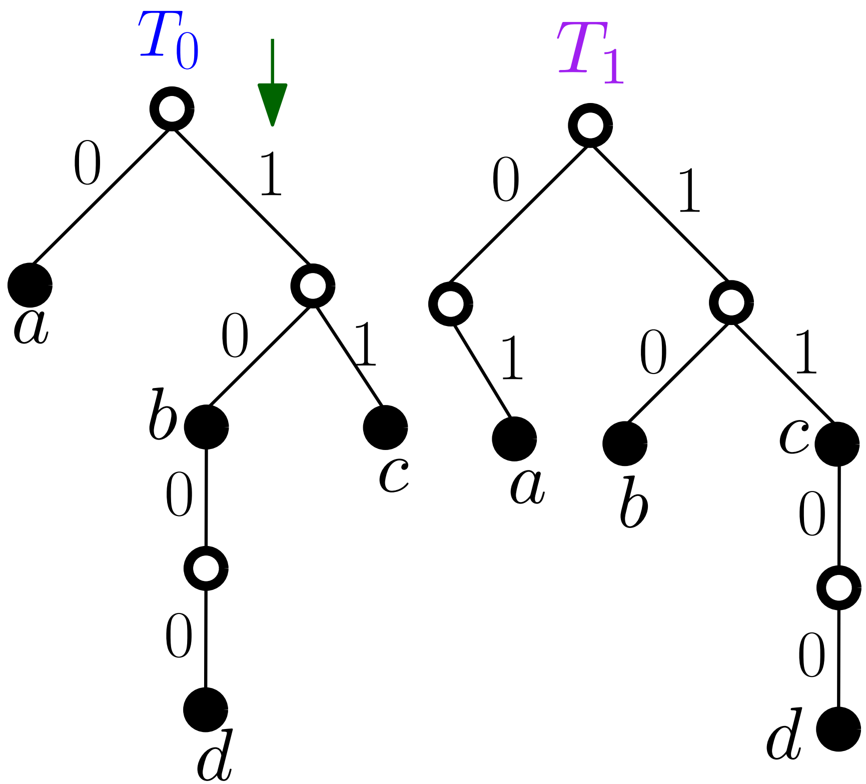
Similar to encoding,
if last symbol decoded used master, use T_1 for next symbol; otherwise use T_0

Example: Decoding 1000010110110



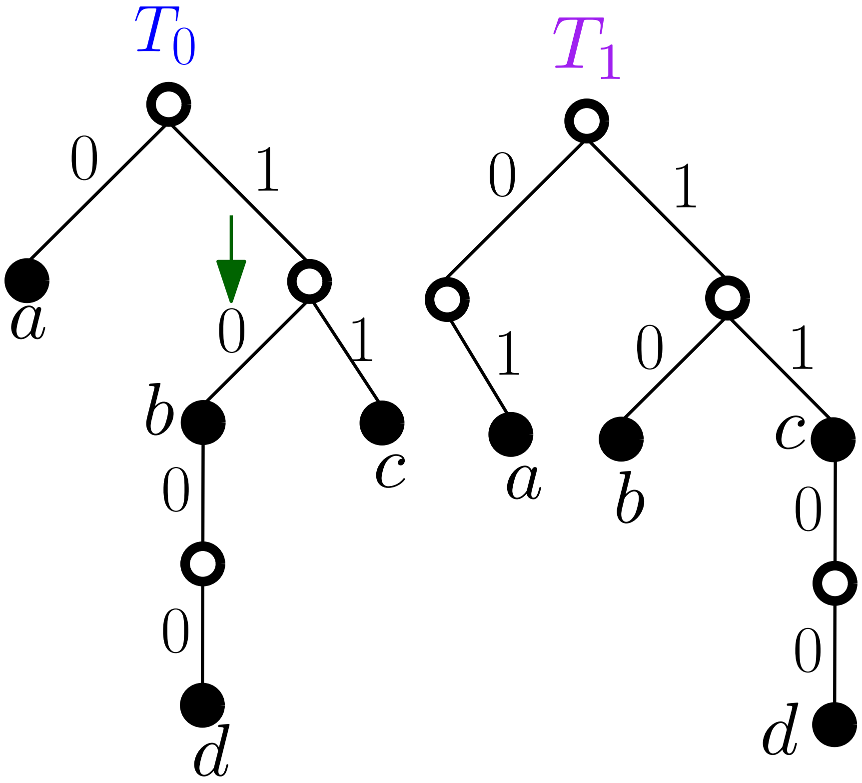
10000**10**1**1**0**1**110

Example: Decoding 1000010110110



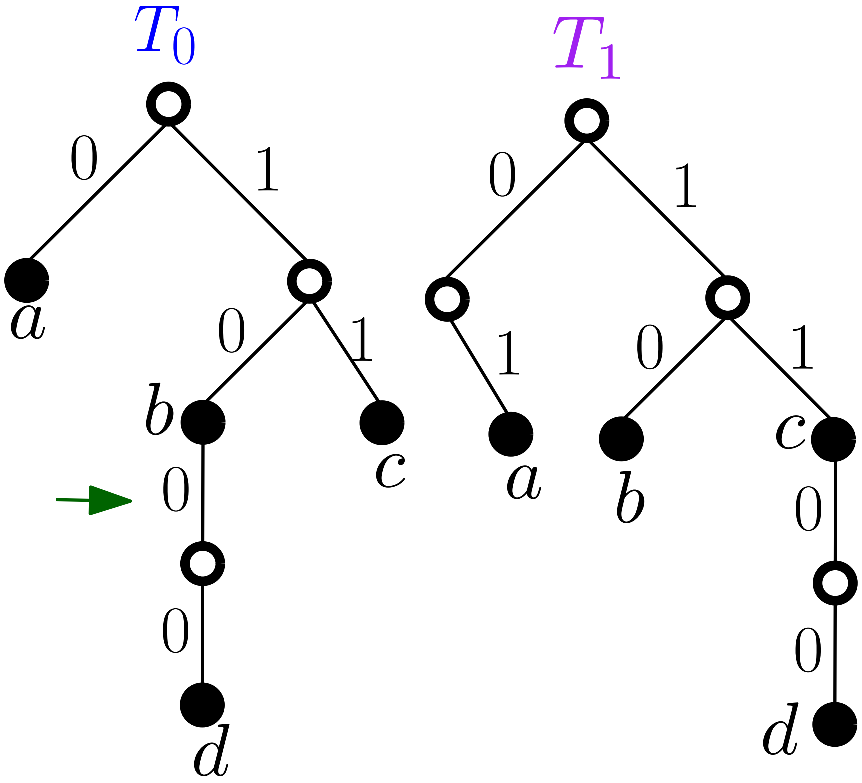
10000**10**11**01**10

Example: Decoding 1000010110110



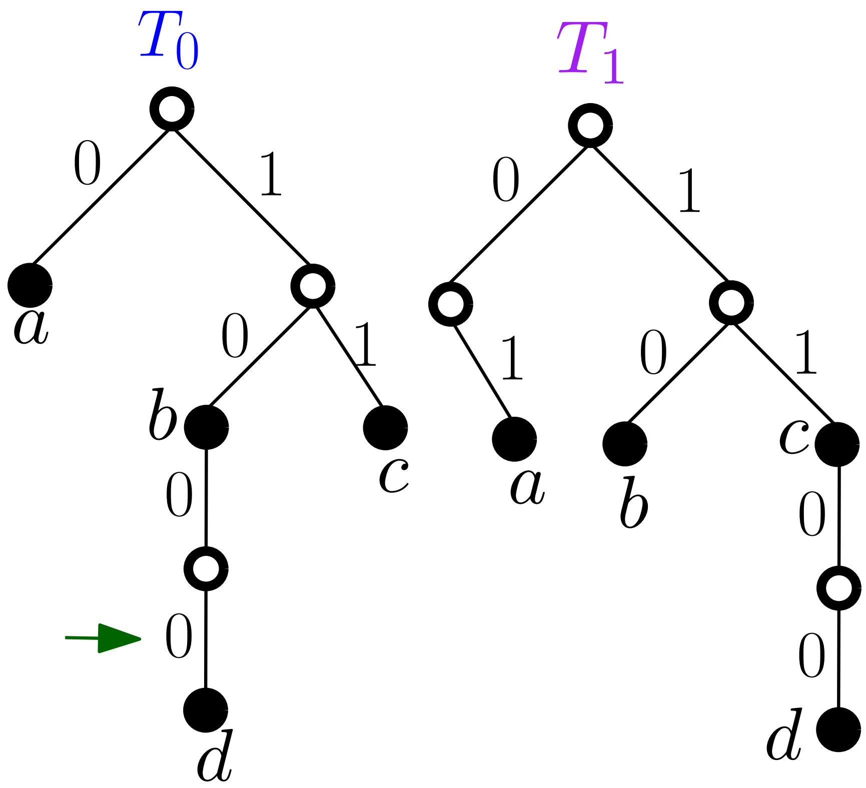
10000**10**1**101**10

Example: Decoding 1000010110110



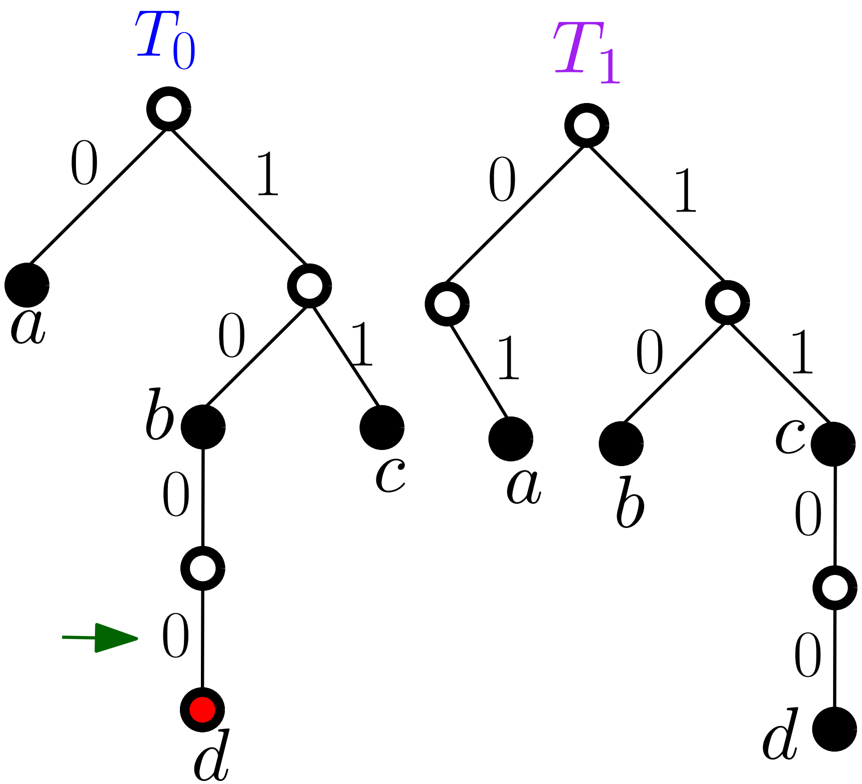
10000**10**11**01**10

Example: Decoding 1000010110110



10000**10**11**01**10

Example: Decoding 1000010110110

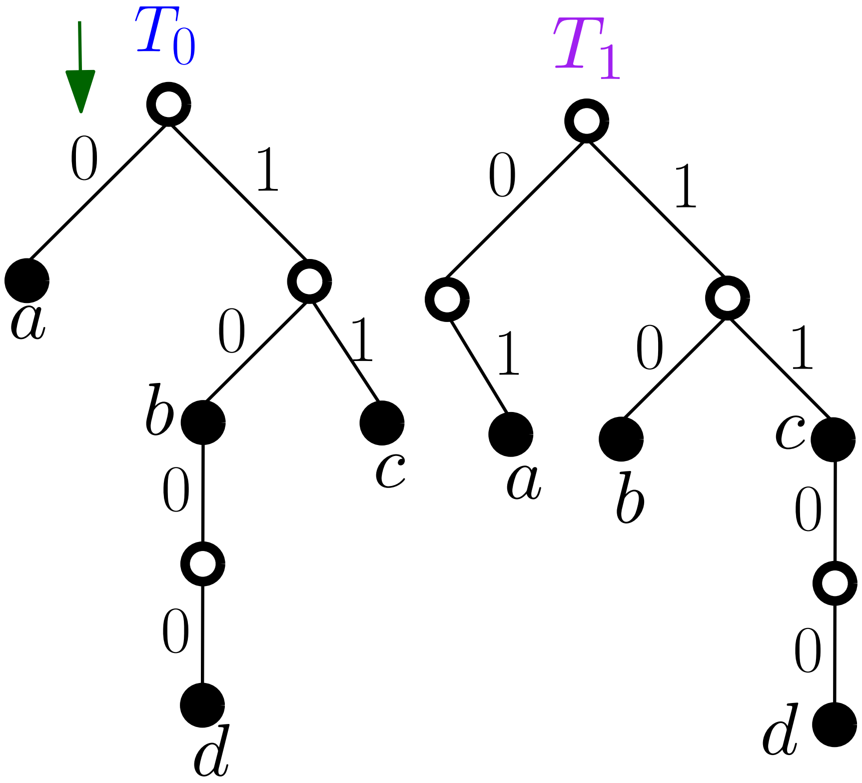


d

1000 | 0**10**11**01**10

Decode d.
Since d is not master, remain in T_0

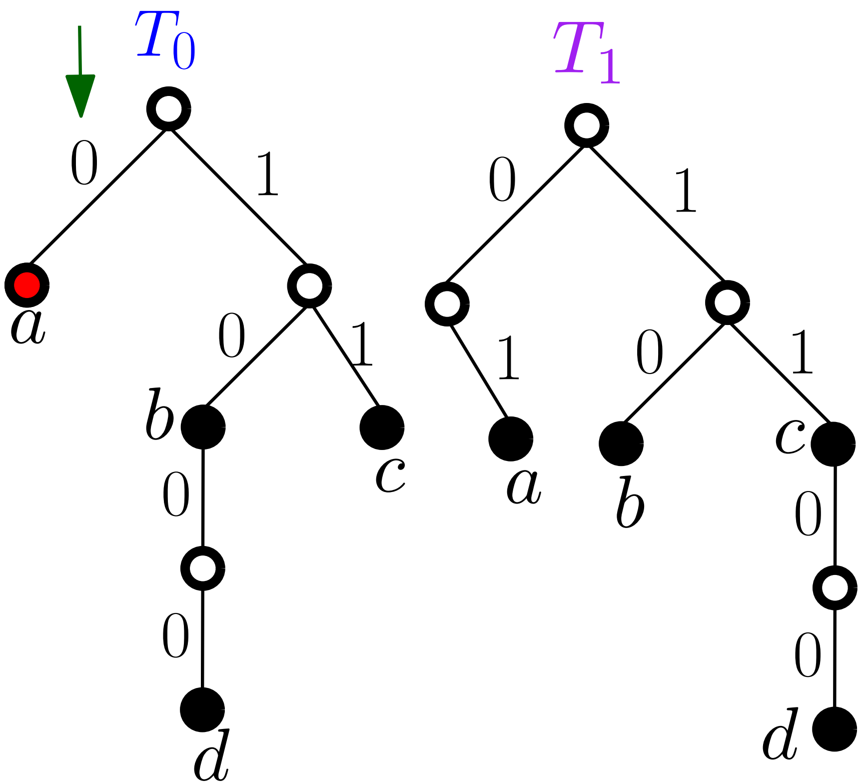
Example: Decoding 1000010110110



d

1000|0**10**11**01**10

Example: Decoding 1000010110110

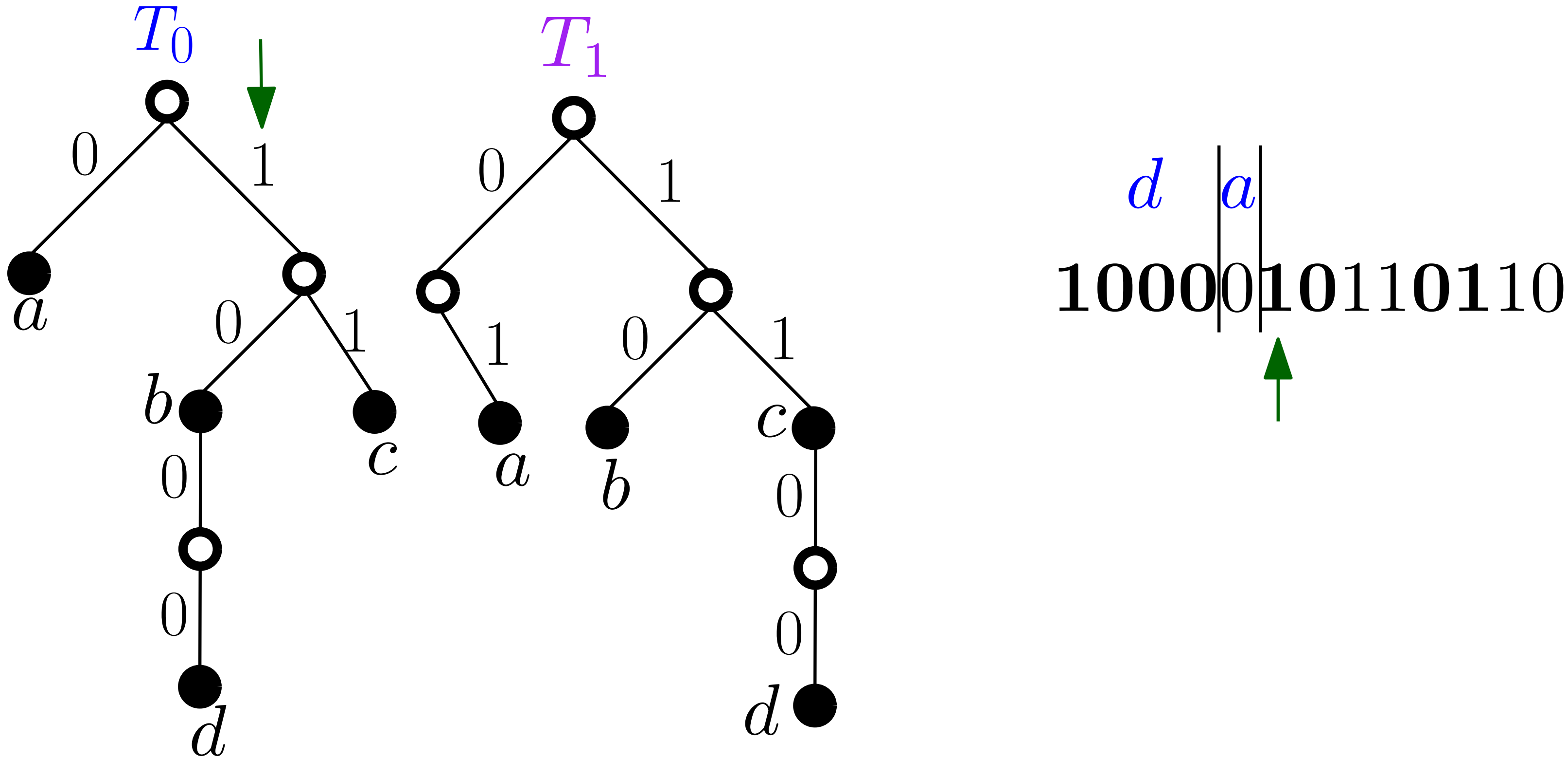


d | a

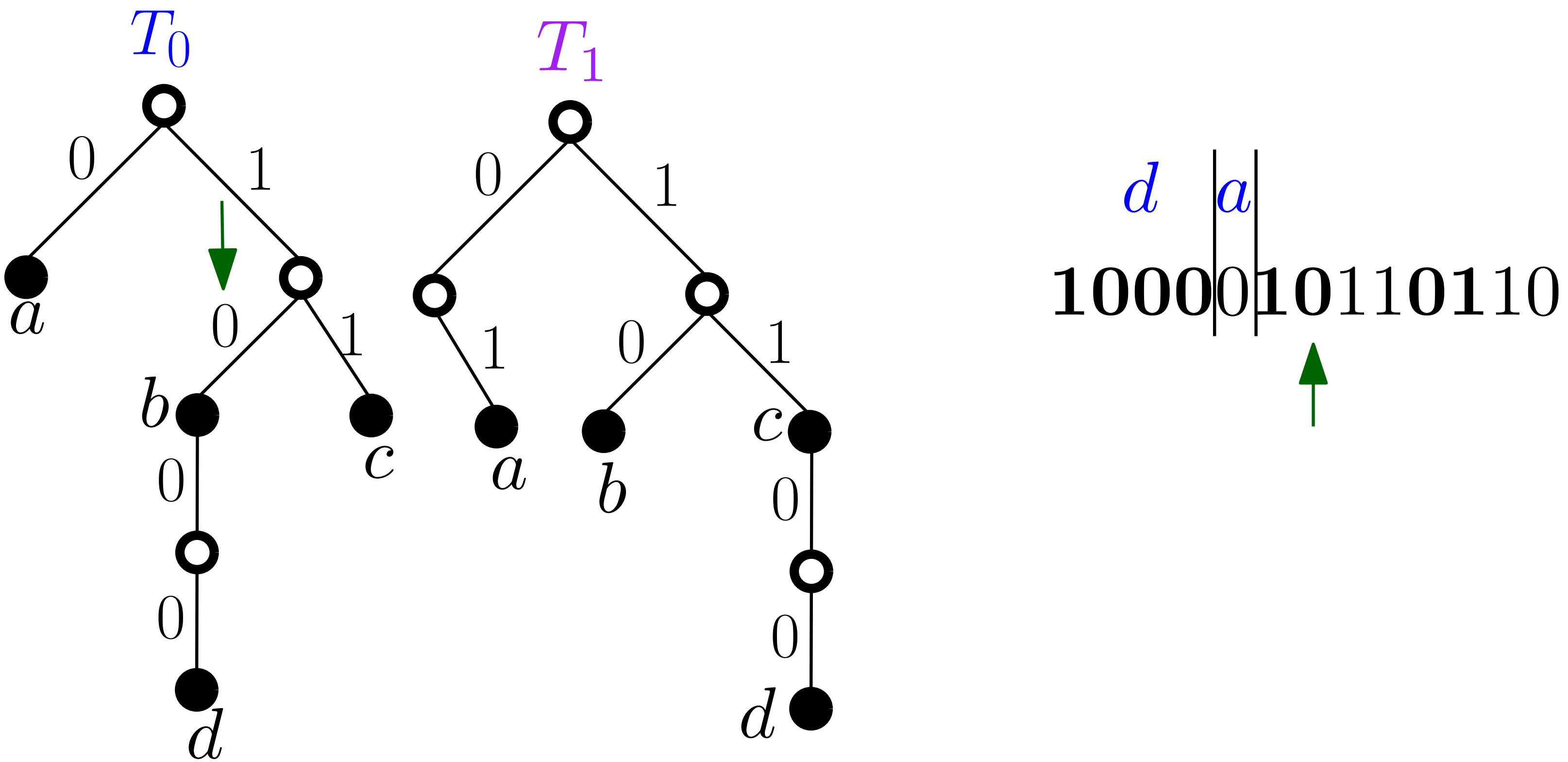
1000|0|**10**11**01**10

Decode a.
Since a is not master,
remain in T_0

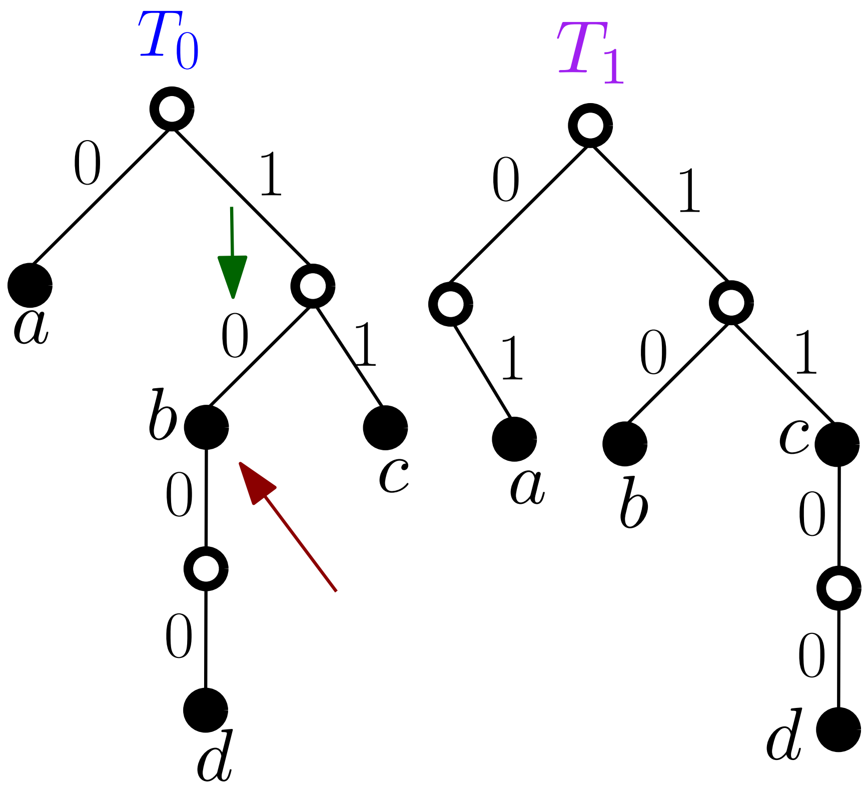
Example: Decoding 1000010110110



Example: Decoding 1000010110110



Example: Decoding 100001011010110



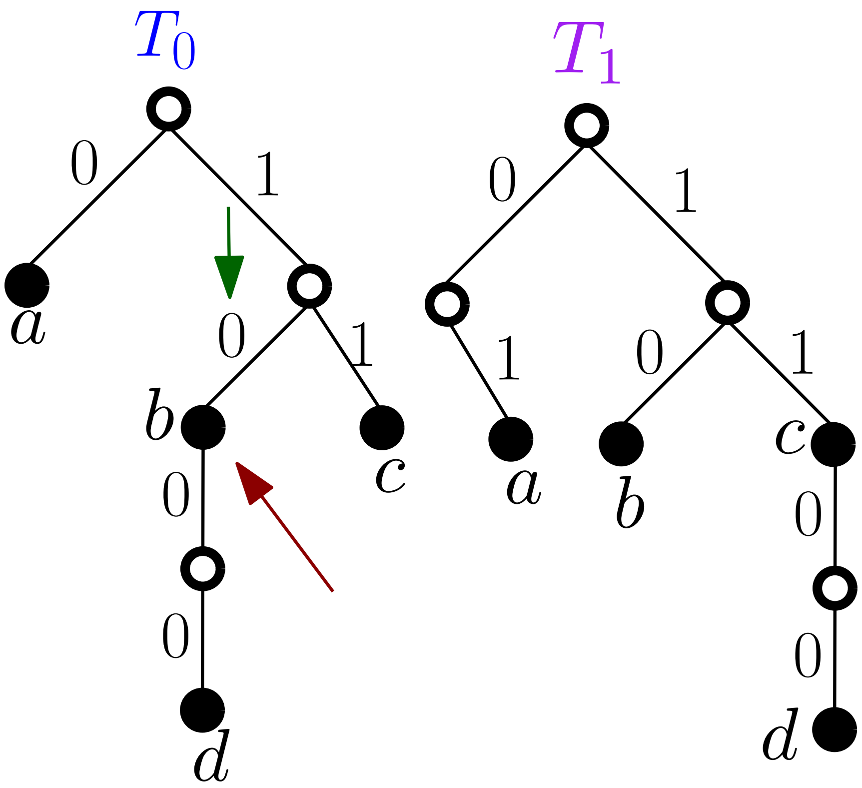
d | a

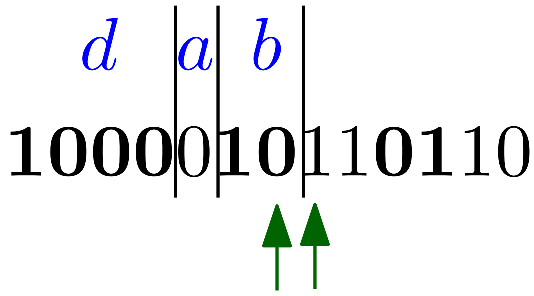
10000|**10**11**01**10

Trace is blocked.

Codeword has 1, but code tree only has 0 edge. Must use master node b.

Example: Decoding 1000010110110



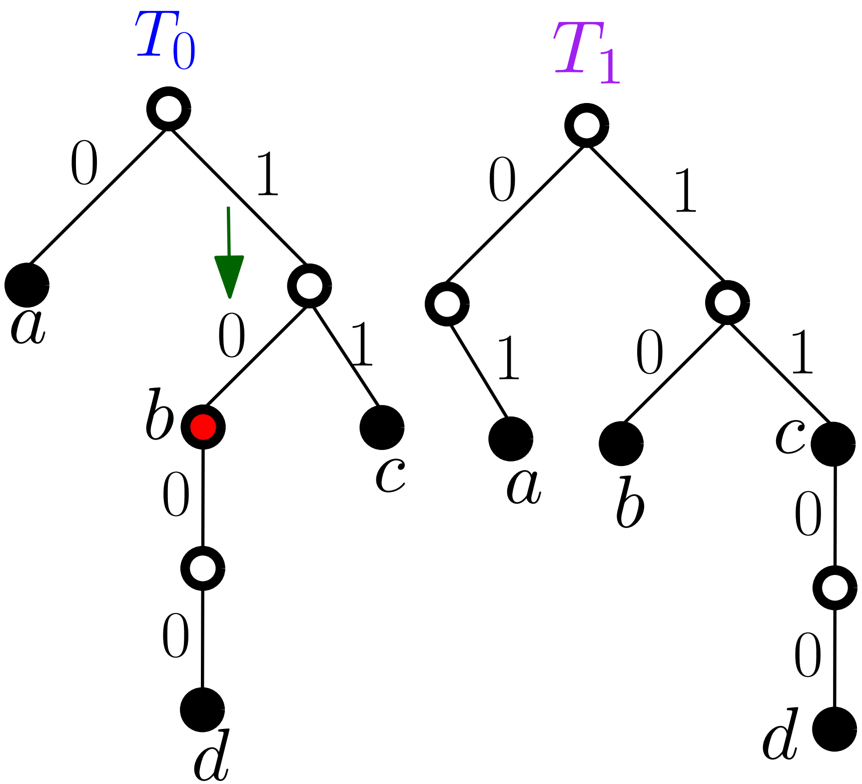


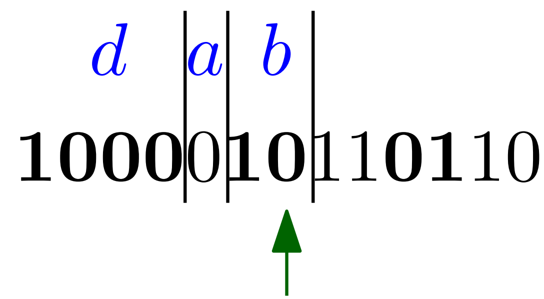
Trace is blocked.

Codeword has 1, but code tree only has 0 edge.

Must use master node b.

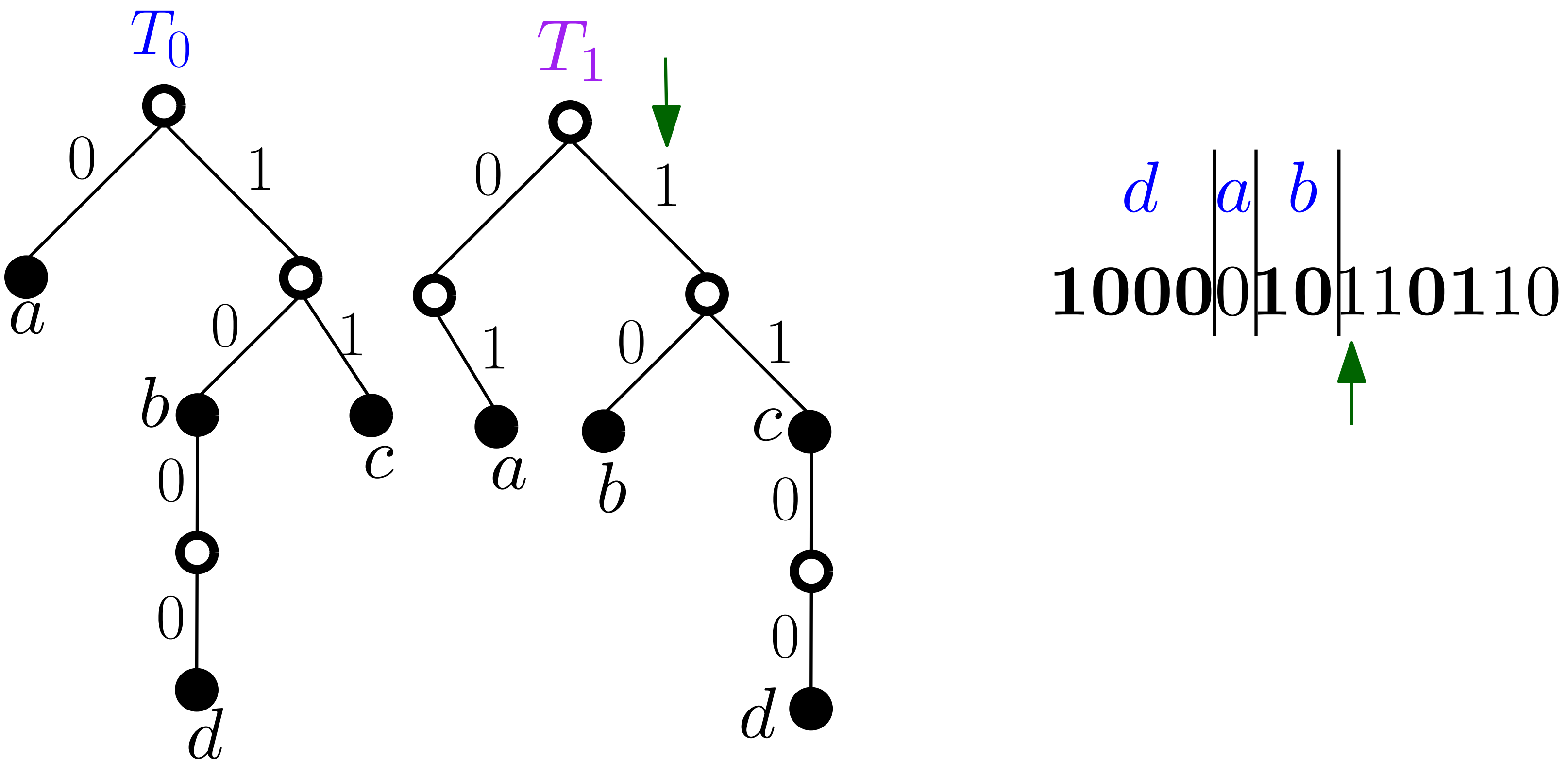
Example: Decoding 1000010110110



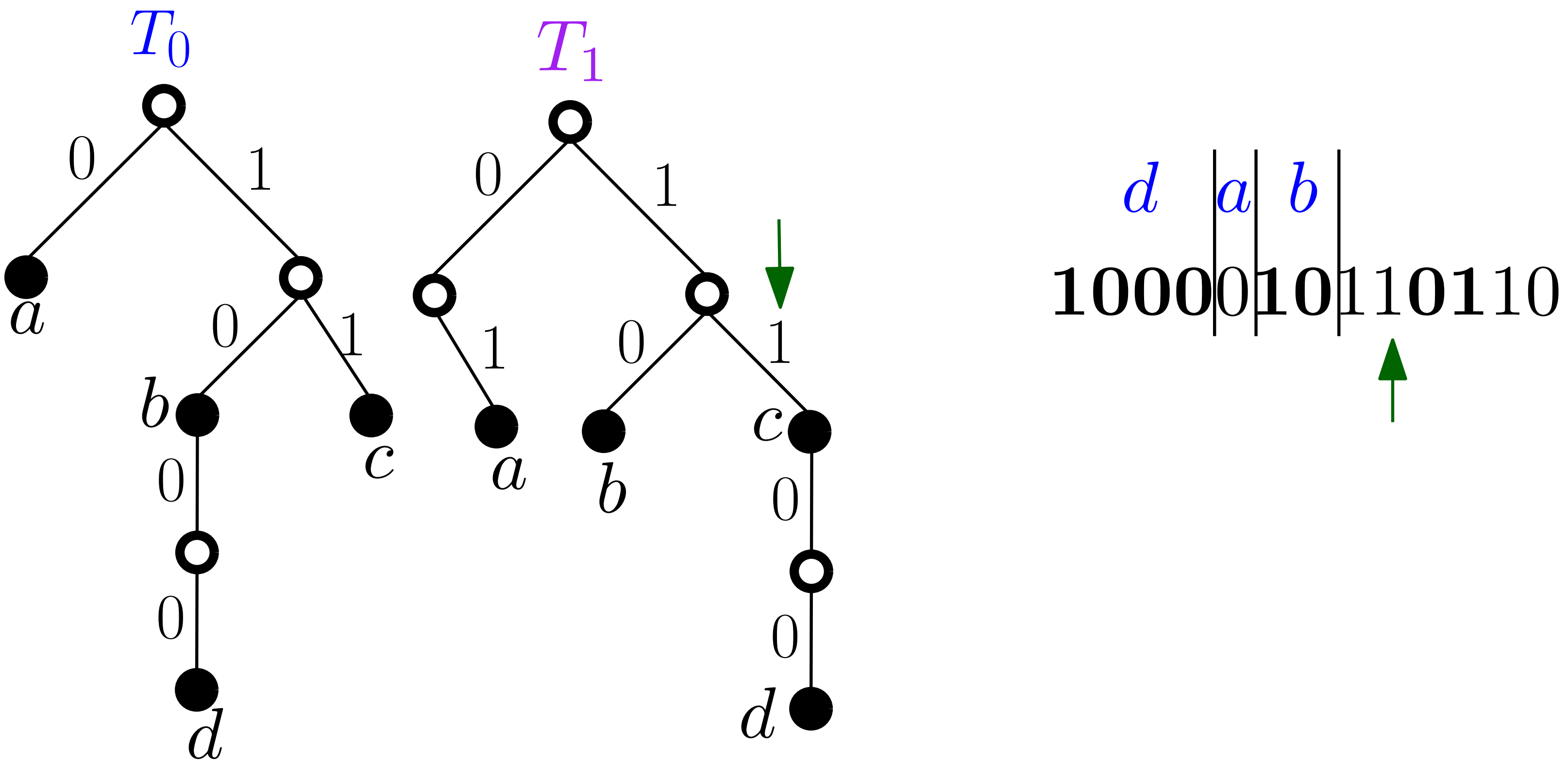


Since b is a master node, switch to T_1.

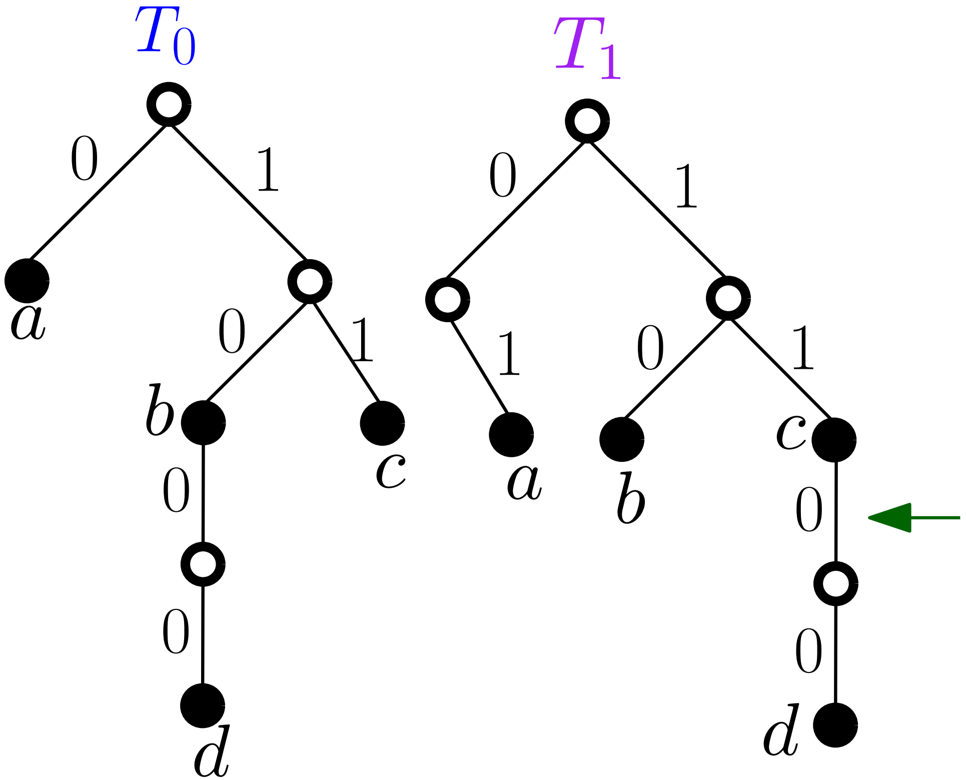
Example: Decoding 1000010110110

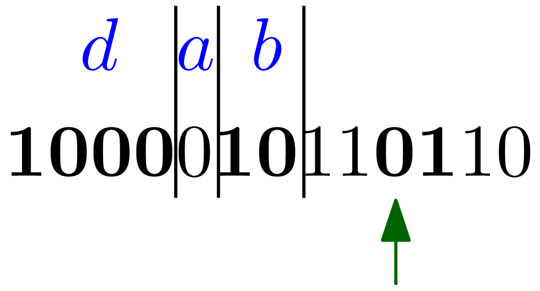


Example: Decoding 1000010110110

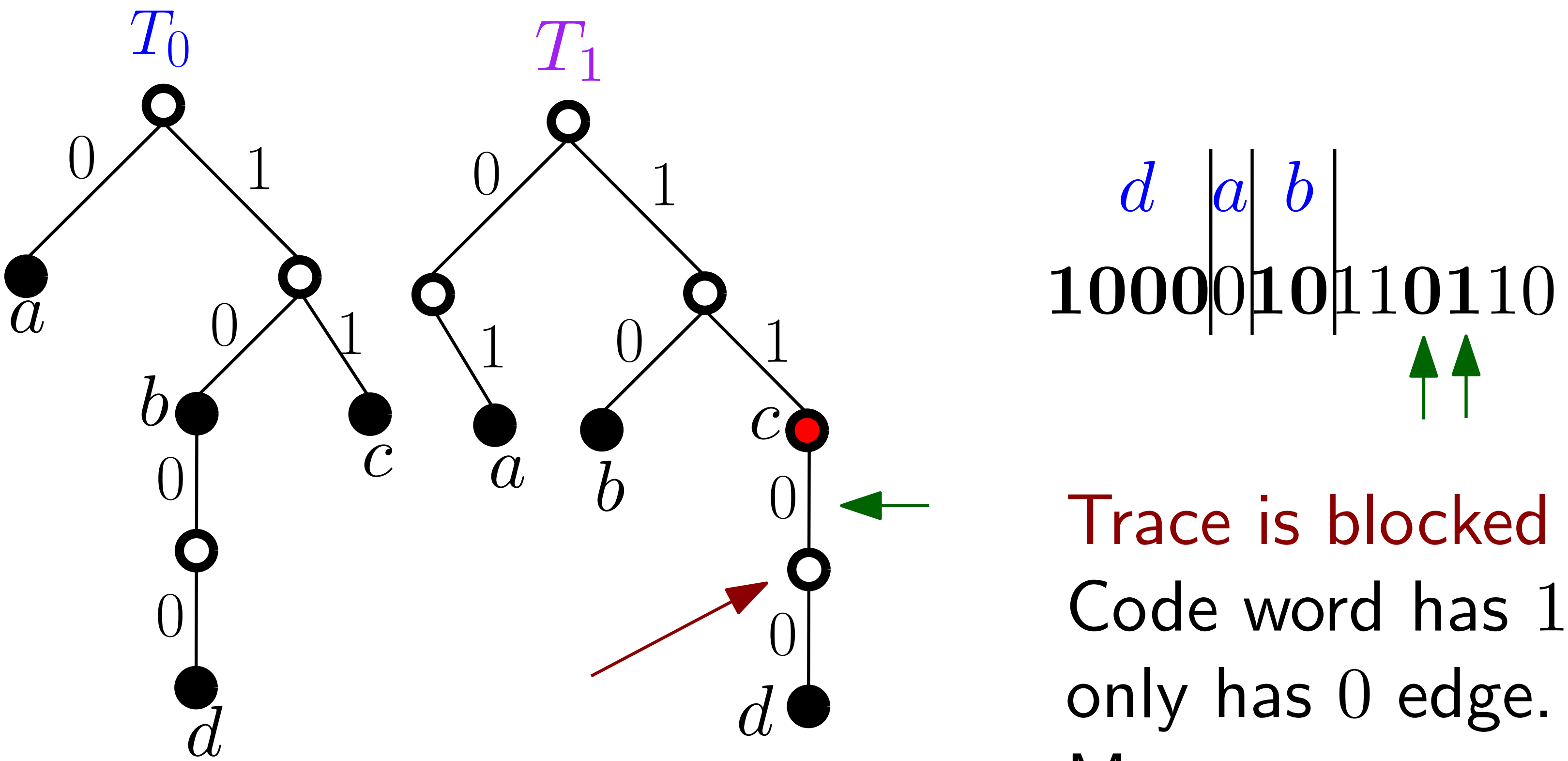


Example: Decoding 1000010110110





Example: Decoding 100001011010

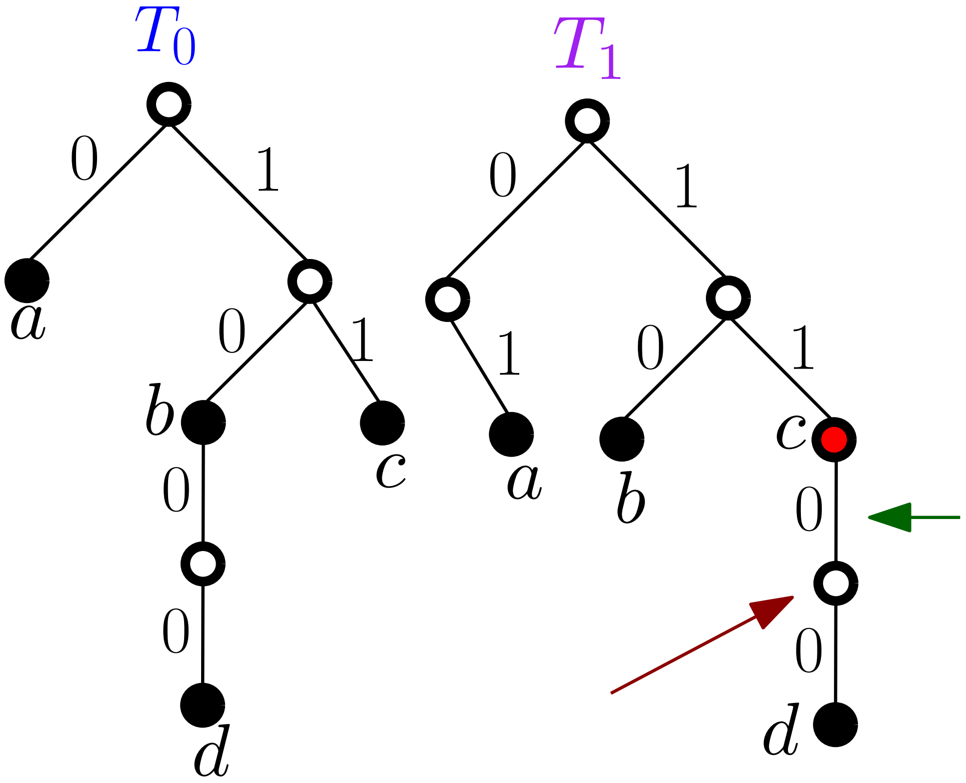


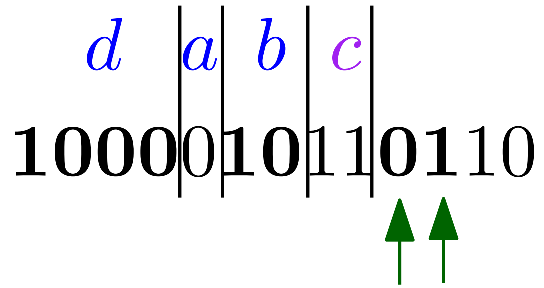
Trace is blocked again.

Code word has 1 but tree only has 0 edge.

Must use master node c.

Example: Decoding 100001011010110



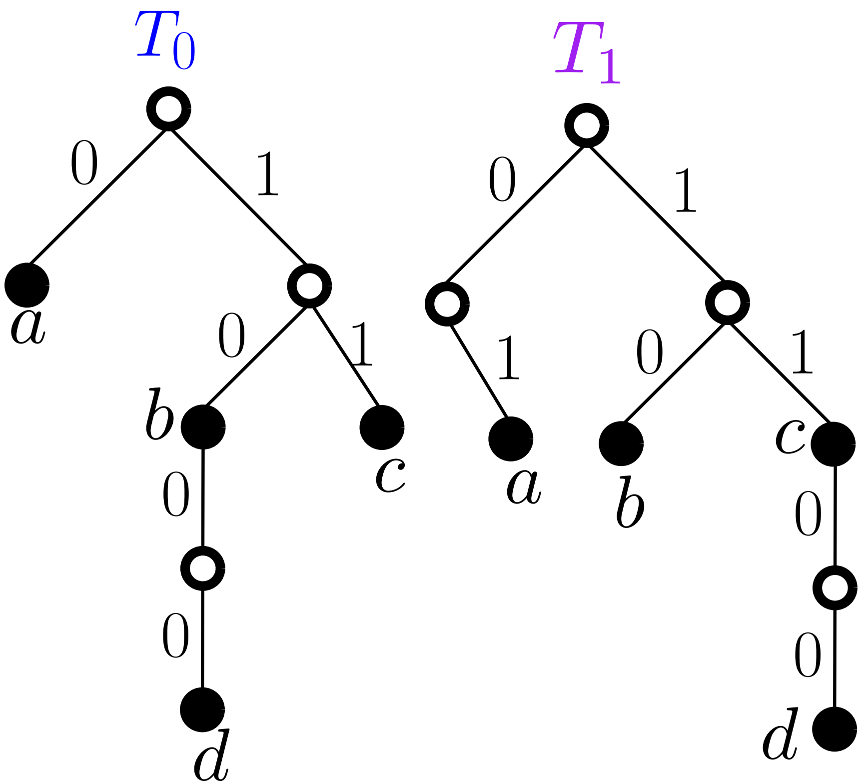


Trace is blocked again.

Code word has 1 but tree only has 0 edge.

Must use master node c.

Example: Decoding 1000010110110

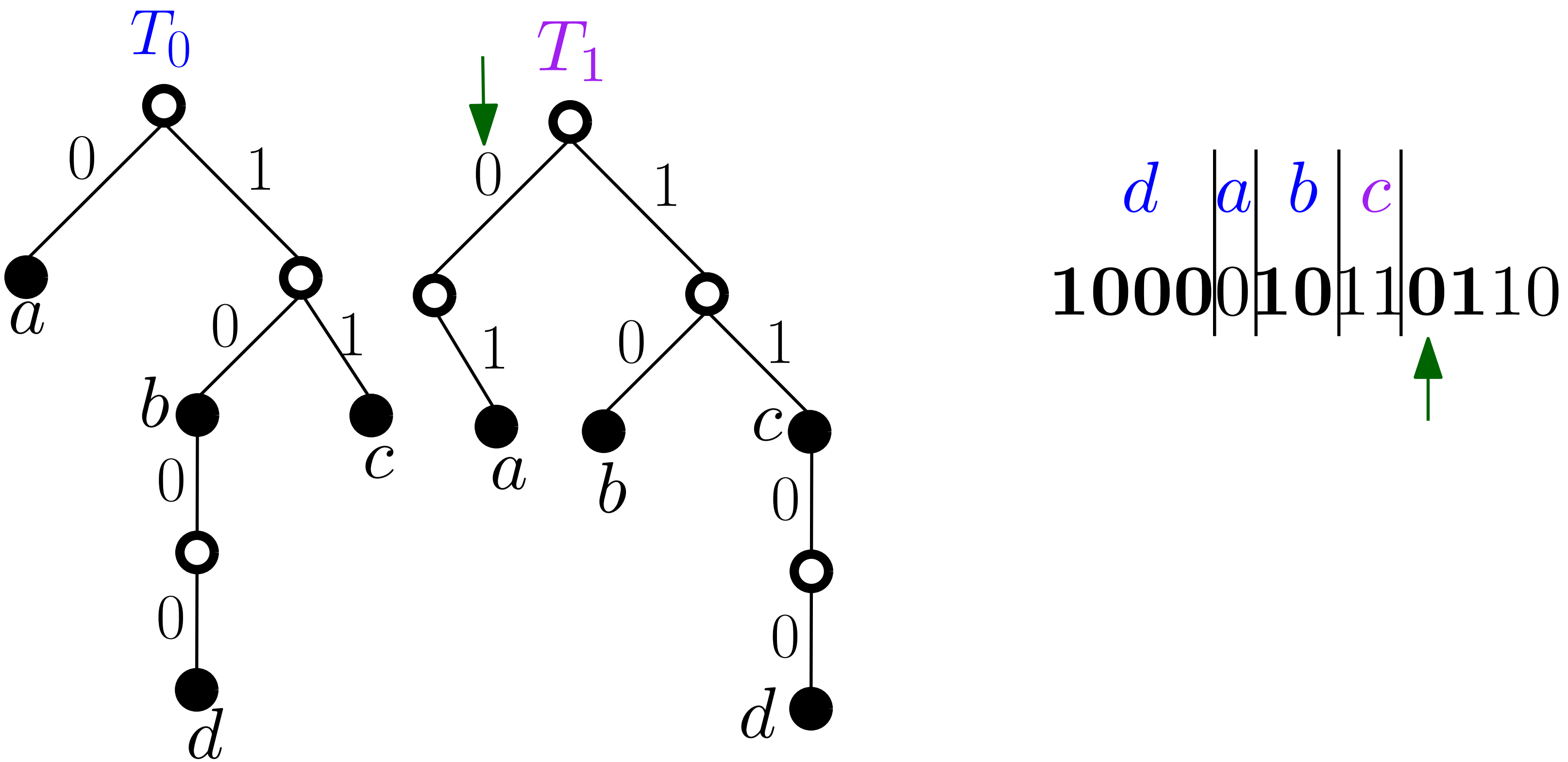


d | a | b | c |

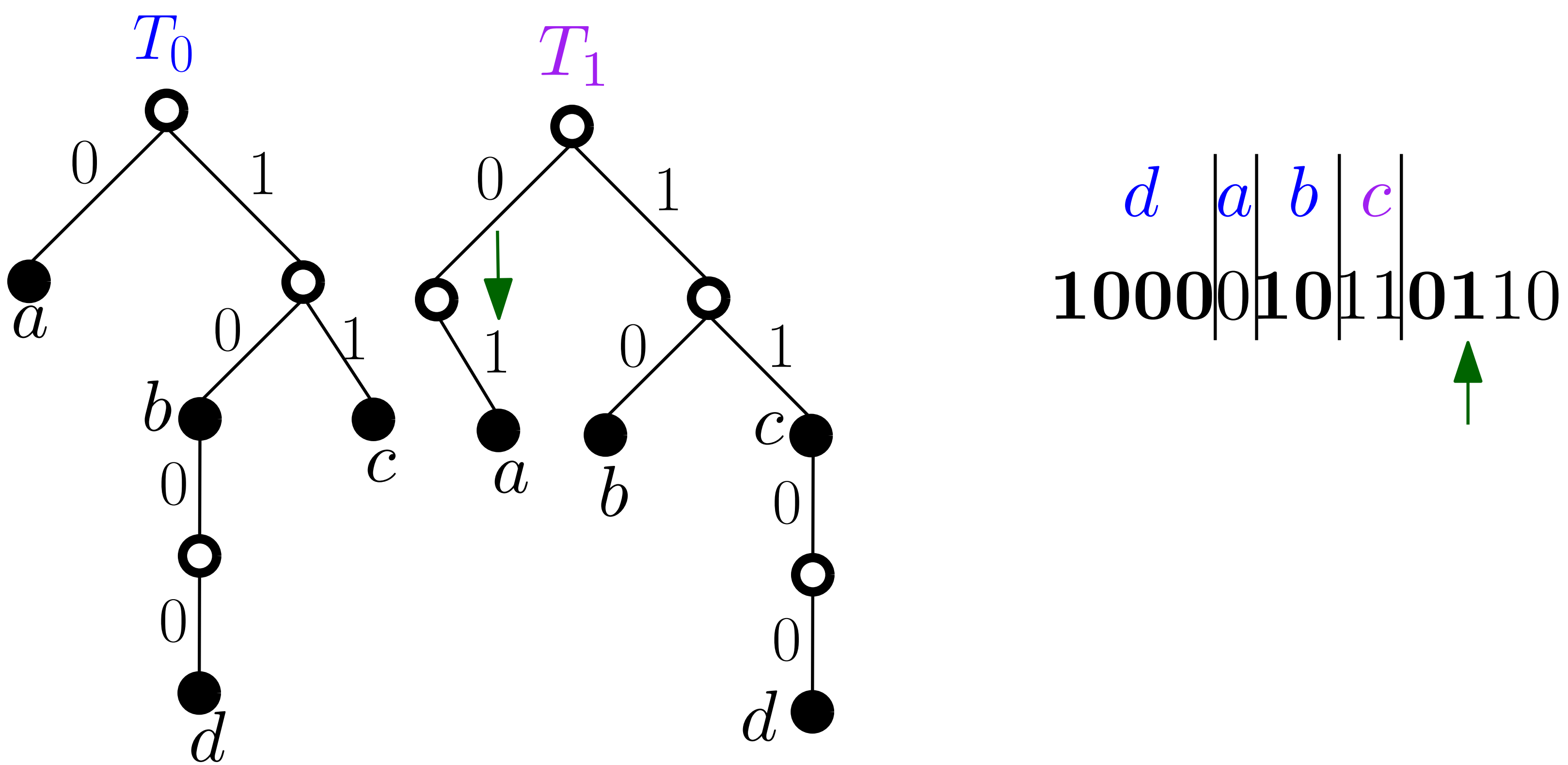
1000|0|**10**|11|**01**10

Since c is a master node, remain in T_1.

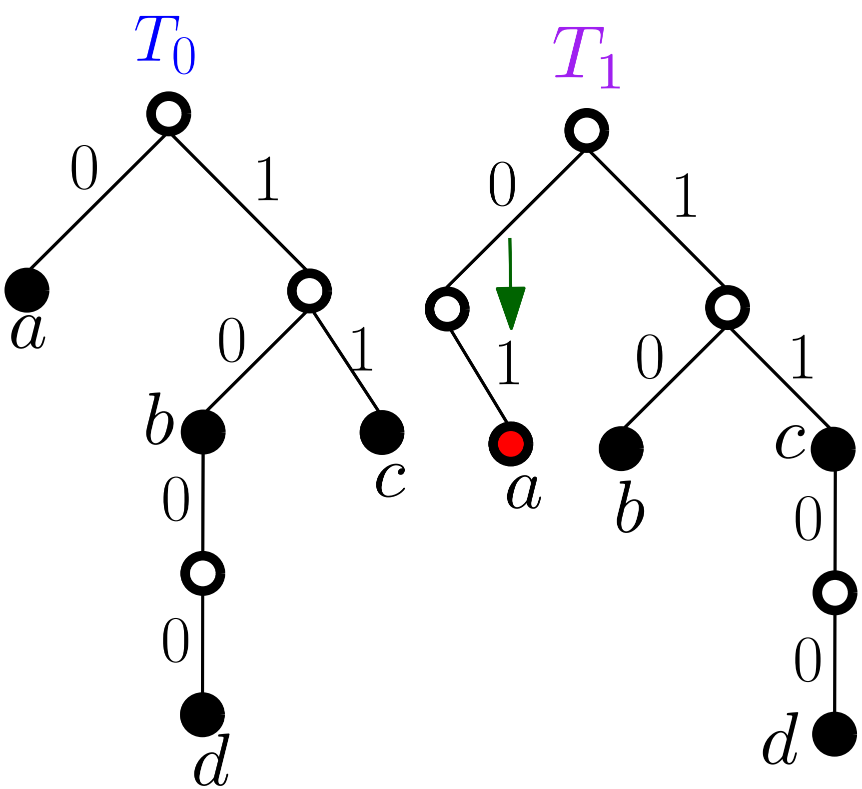
Example: Decoding 1000010110110



Example: Decoding 1000010110110



Example: Decoding 100001011010110

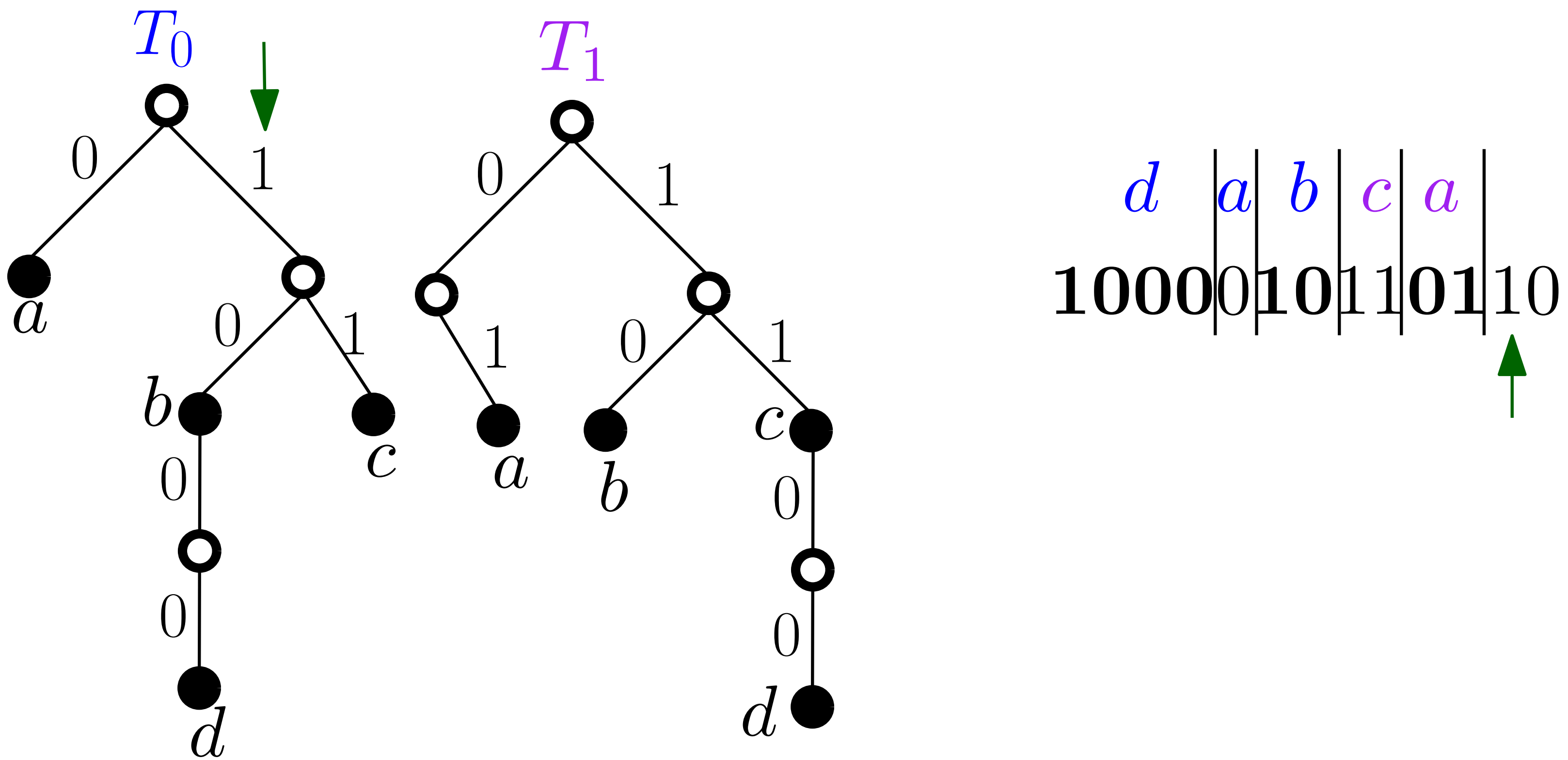


d | a | b | c | a |

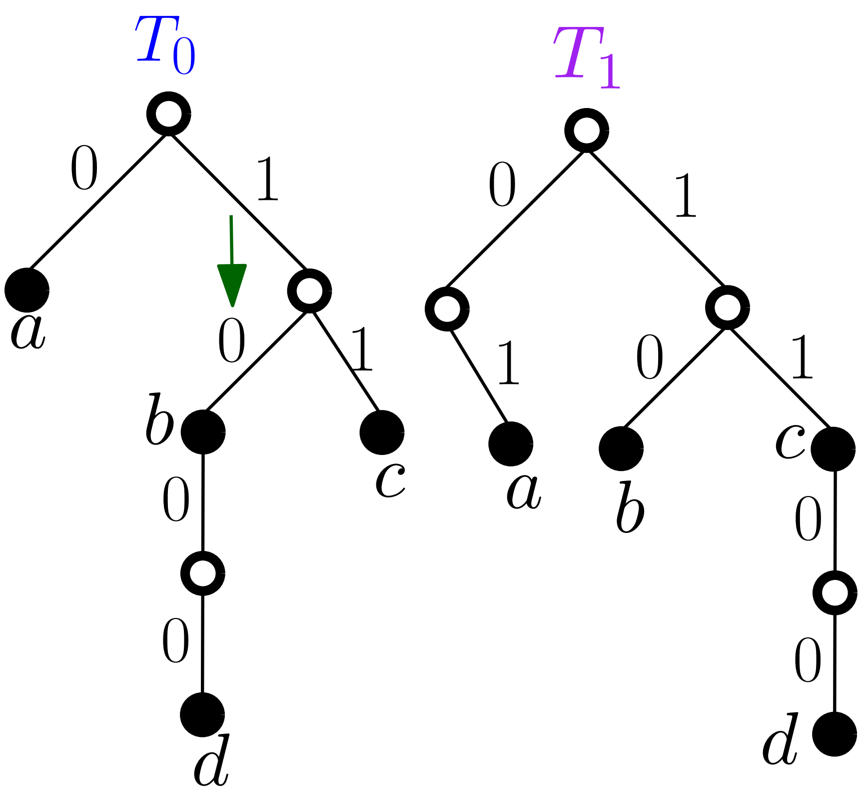
1000|0|**10**|11|**01**|10

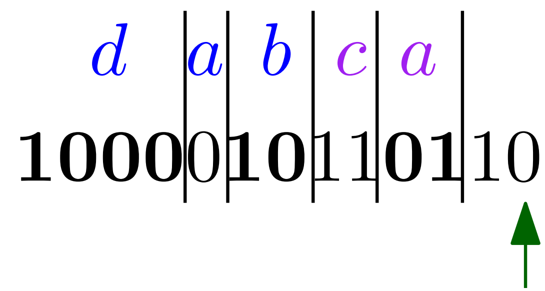
Decode a.
Since a is not master,
switch to T_0

Example: Decoding 1000010110110

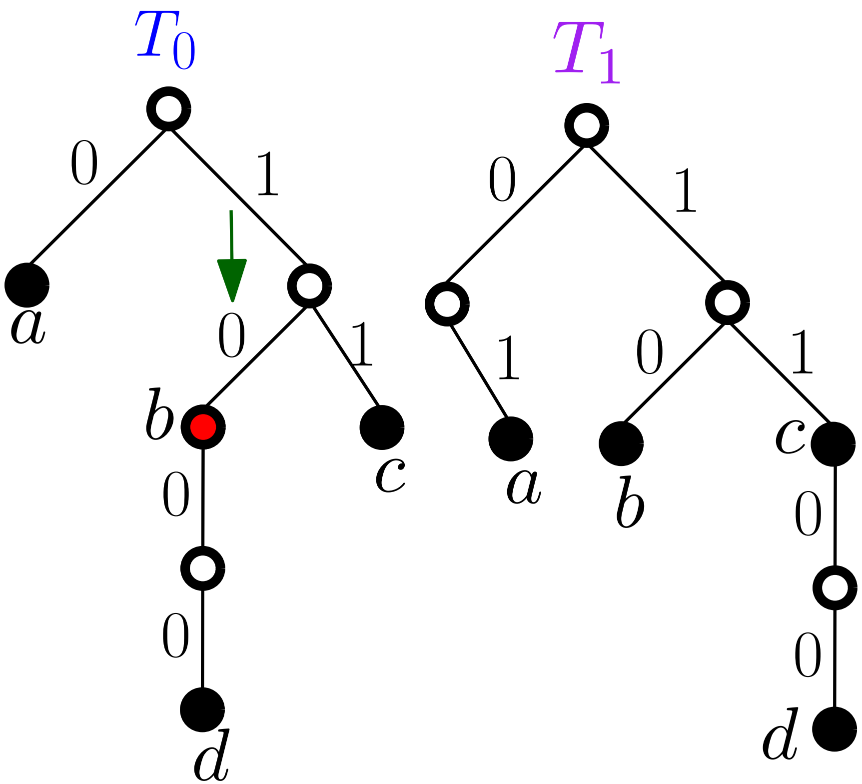


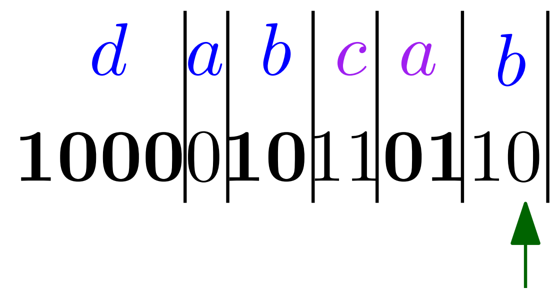
Example: Decoding 1000010110110





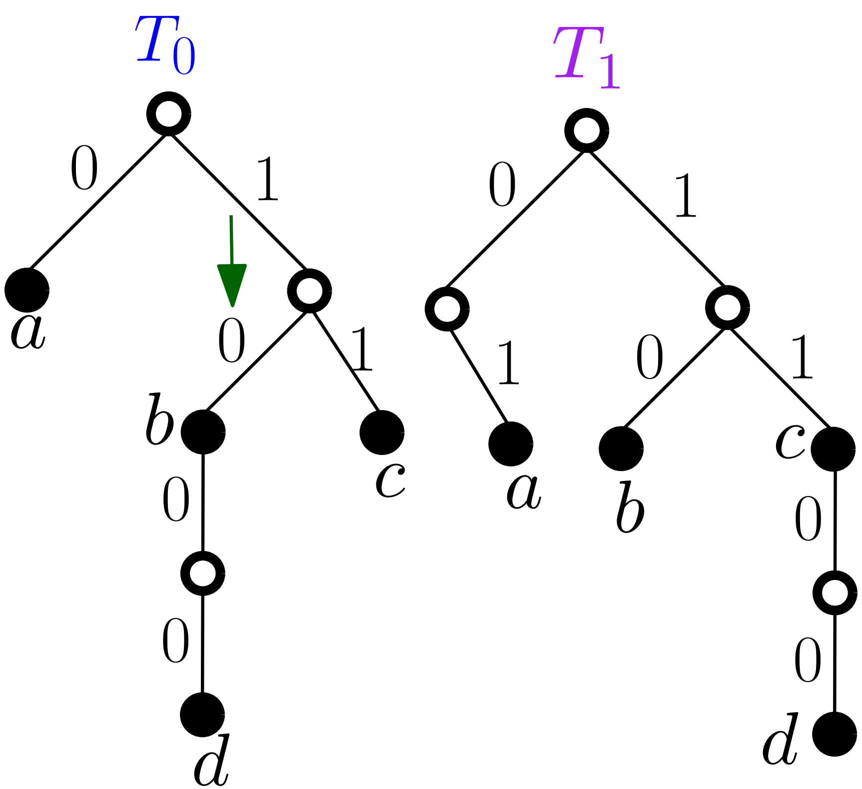
Example: Decoding 100001011010110





Decode b

Example: Decoding 100001011010110



d	a	b	c	a	b
1000	0	**10**	11	**01**	10

The final decoded word is $dabcab$

- Optimal AIFV-2 Codes compress at least as well as Huffman coding. There are examples (such as the last example, calculation later) that can be shown to beat Huffman compression.

- Allowing a decoding delay of 2 bits, and 2 trees permits improving the compression.

- Optimal AIFV-2 Codes compress at least as well as Huffman coding. There are examples (such as the last example, calculation later) that can be shown to beat Huffman compression.

- Allowing a decoding delay of 2 bits, and 2 trees permits improving the compression.

- Constructing Optimal Huffman Codes is $O(n \log n)$, or $O(n)$ if the probabilities are sorted.

- Constructing Optimal AIFV-2 codes is much more difficult. State of the art had no polynomial algorithm.

References and Extensions

General AIFV References

(1) H. Yamamoto and X. Wei,
" Almost instantaneous FV codes," *2013 IEEE ISIT*

(2) W. Hu, H. Yamamoto, and J. Honda,
"Worst-case redundancy of optimal binary AIFV codes and their extended codes," *IEEE Transactions on Information Theory*, 2017

(3) H. Yamamoto, M. Tsuchihashi, and J. Honda,
" Almost instantaneous Fixed-to-variable length codes,
IEEE Transactions on Information Theory. 2015

AIFV-m Codes (a generalization to m coding trees)

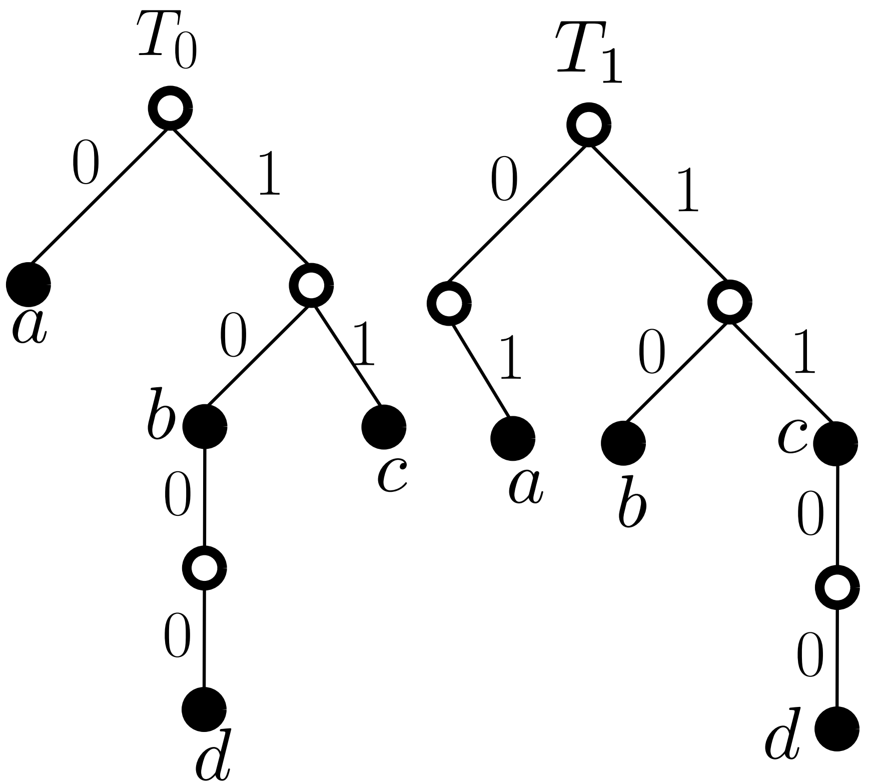
(4) H. Yamamoto and K. Iwata,
"An iterative algorithm to construct optimal binary AIFV-m codes,"
IEEE ITW'17

(5) K. Iwata and H. Yamamoto, "A dynamic programming algorithm to construct optimal code trees of AIFV codes," *ISITA'16,*

Outline

- Introduction
- AIFV-2 codes: cost and algorithm
- A Geometric Interpretation of the old algorithm
    - A New Binary Search Algorithm
    - An Ellipsoid Algorithm
- Extensions to AIFV-k codes (skip)
- Summing up and open questions

Calculating average code length $L_{AIFV}(T_0, T_1)$

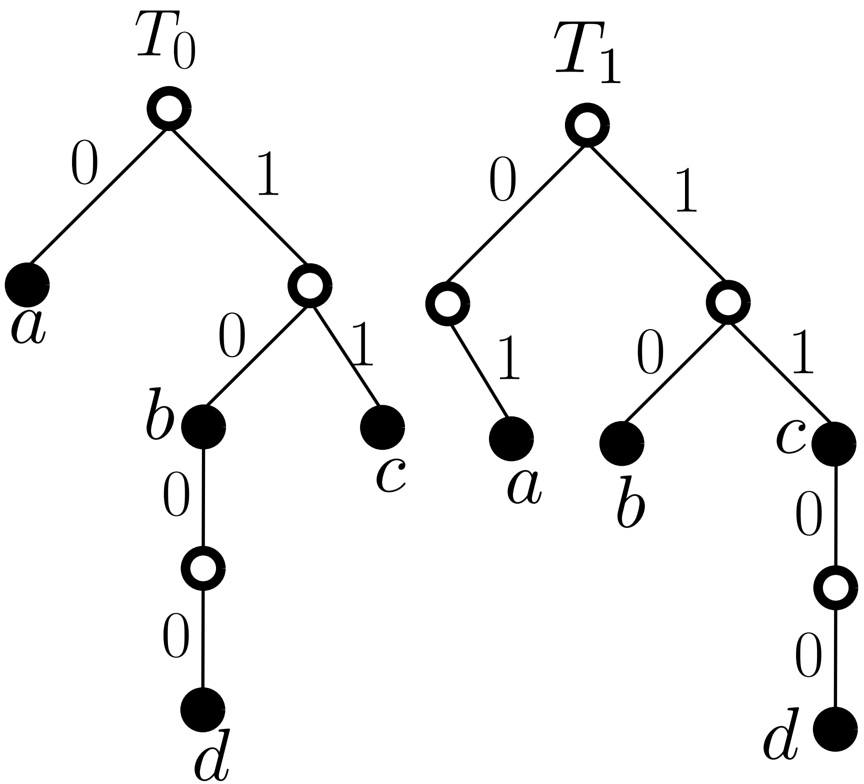


$\forall x \in \mathcal{X}$, let $c_s(x)$ be the code word representing x in T_s.

The *average length* of individual code tree T_s is

$$L(T_s) = \sum_{x \in \mathcal{X}} |c_s(x)| p_x$$

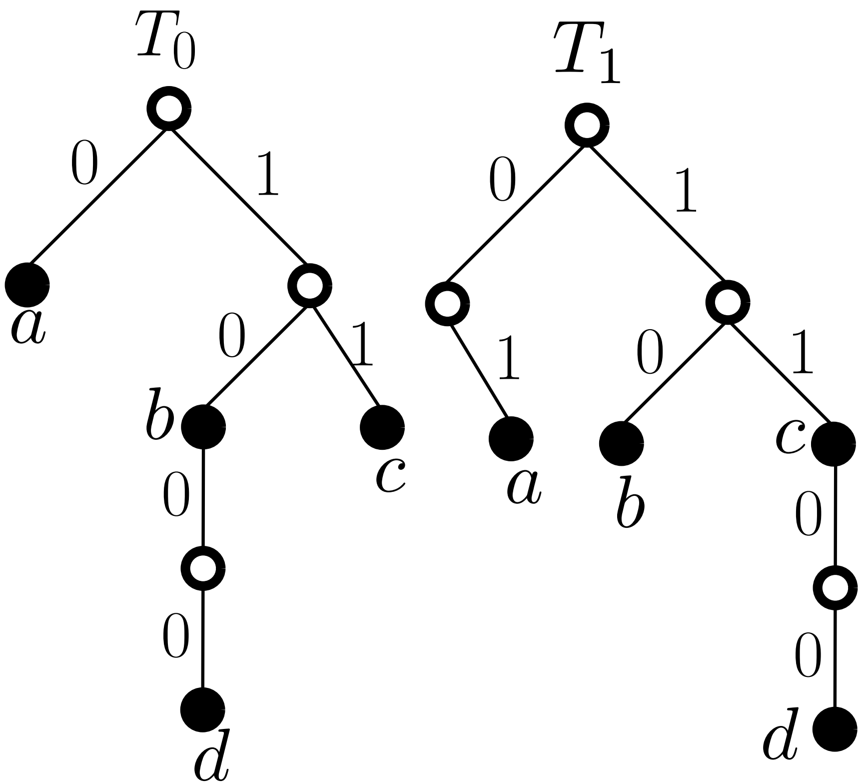
Calculating average code length $L_{AIFV}(T_0, T_1)$



Fix T_0, T_1.
Consider randomly generated string $S = s_1, s_2, \ldots, \in \mathcal{X}^*$.

The tree used to encode s_i is modelled by a two state ergodic Markov Chain.

Calculating average code length $L_{AIFV}(T_0, T_1)$

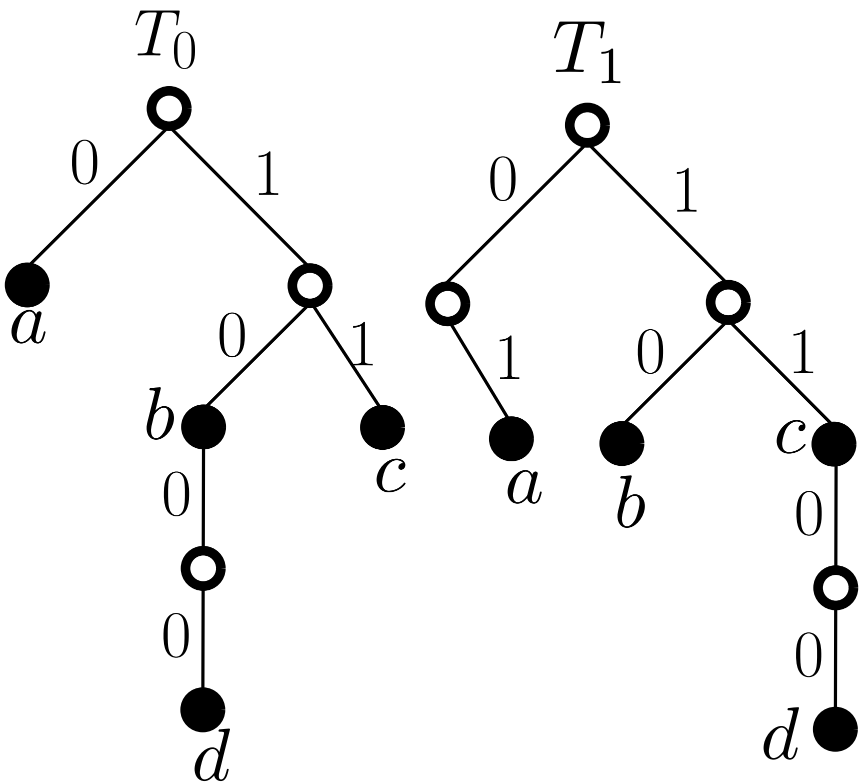


Fix T_0, T_1.
Consider randomly generated string $S = s_1, s_2, \ldots, \in \mathcal{X}^*$.

The tree used to encode s_i is modelled by a two state ergodic Markov Chain.

Let $q_0(T_1)$ be sum of leaf weights in T_1; $q_1(T_0)$ the sum of master weights in T_0

Calculating average code length $L_{AIFV}(T_0, T_1)$



Fix T_0, T_1.
Consider randomly generated string $S = s_1, s_2, \ldots, \in \mathcal{X}^*$.

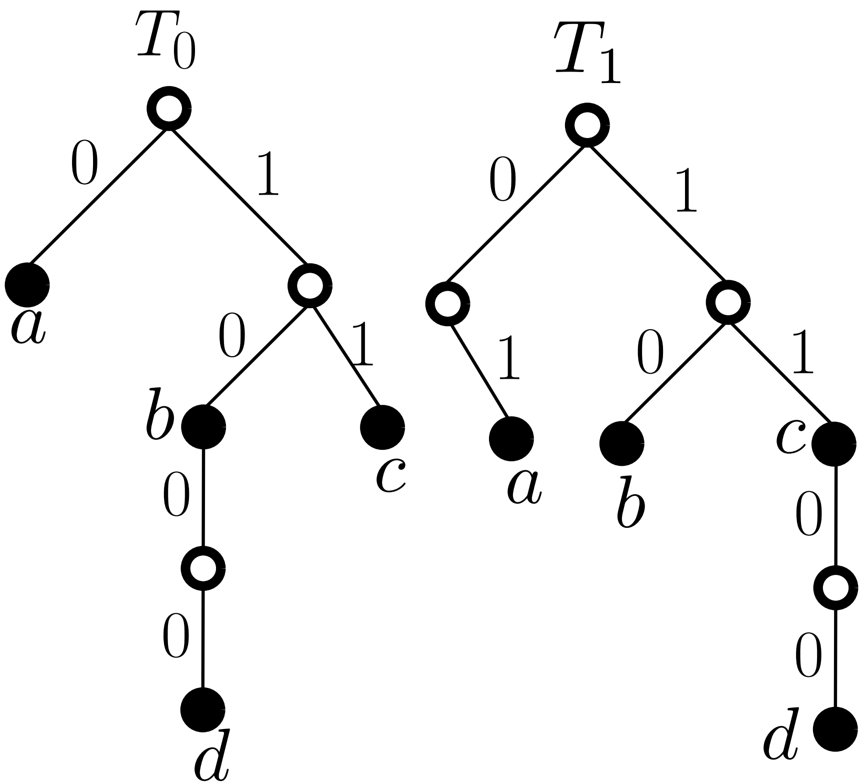
The tree used to encode s_i is modelled by a two state ergodic Markov Chain.

Let $q_0(T_1)$ be sum of leaf weights in T_1; $q_1(T_0)$ the sum of master weights in T_0

Let $s, \hat{s} \in \{0,1\}, s \neq \hat{s}$. Working through the details, the stationary probability of using T_s is given by

$$P(s|T_0, T_1) = \frac{q_s(T_{\hat{s}})}{q_0(T_1) + q_1(T_0)}$$

Calculating average code length $L_{AIFV}(T_0, T_1)$



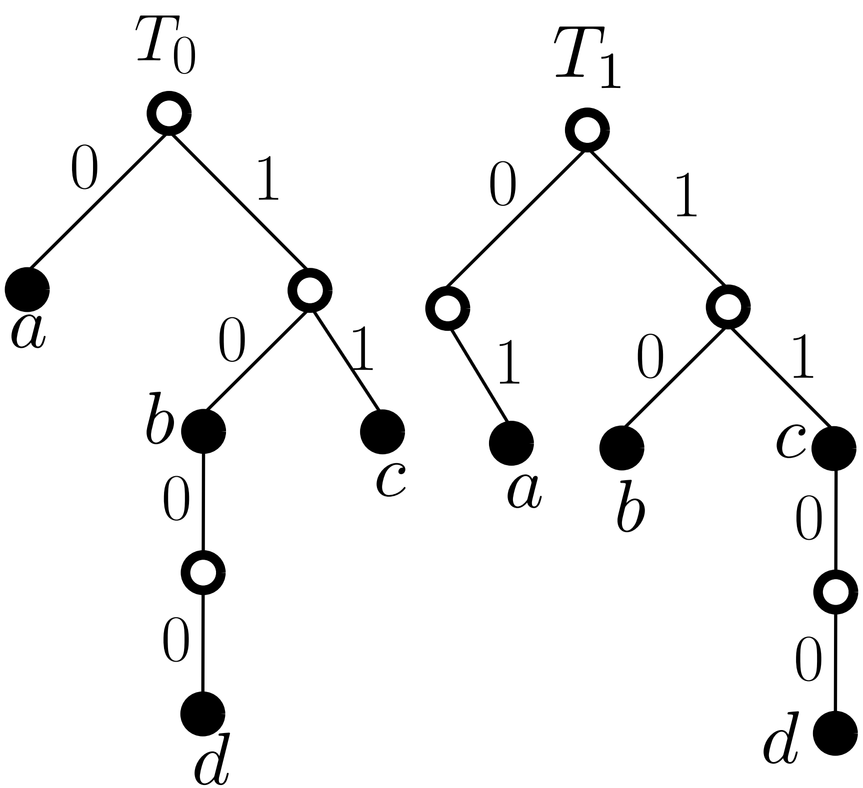
Fix T_0, T_1.
Consider randomly generated string $S = s_1, s_2, \ldots, \in \mathcal{X}^*$.

The tree used to encode s_i is modelled by a two state ergodic Markov Chain.

$$L_{AIFV}(T_0, T_1) = P(0|T_0, T_1)L(T_0) + P(1|T_0, T_1)L(T_1)$$

- $P(0|T_0, T_1)$: stat. prob of being in T_0
- $L(T_0)$: cost of T_0
- $P(1|T_0, T_1)$: stat. prob of being in T_1
- $L(T_1)$: cost of T_1

Calculating average code length $L_{AIFV}(T_0, T_1)$



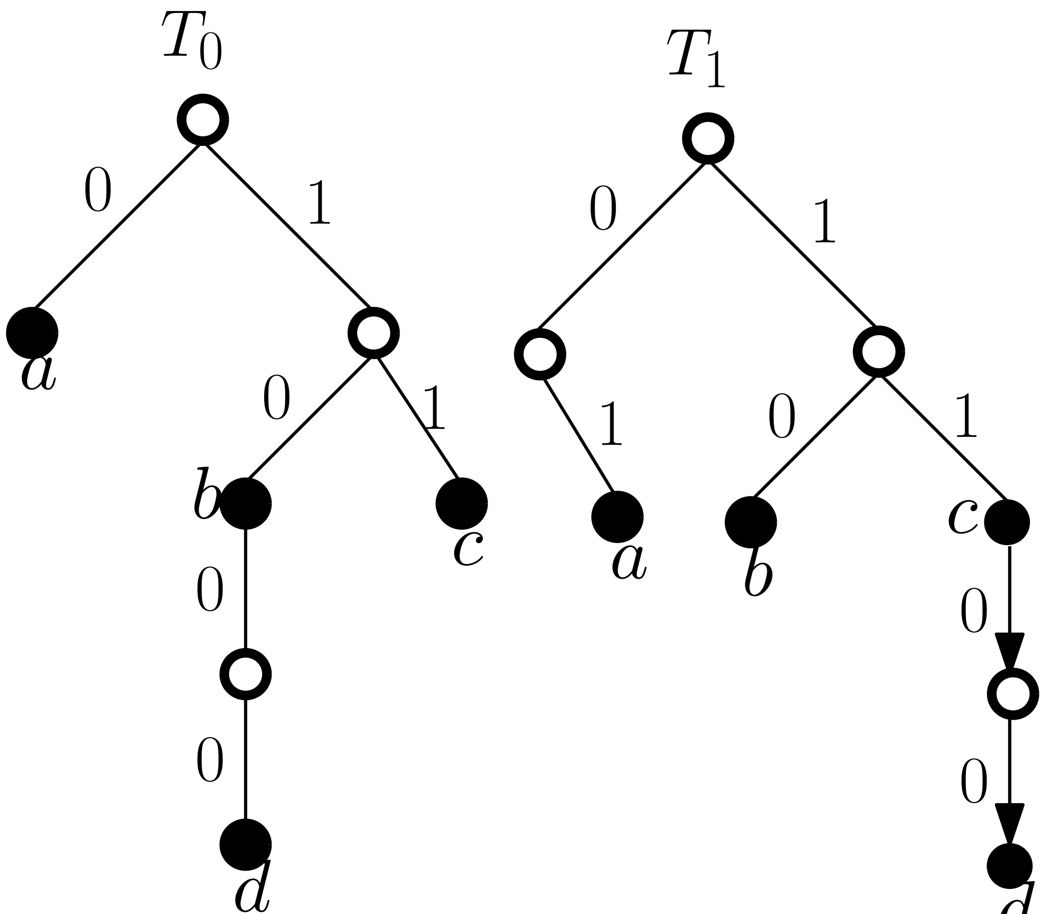
Fix T_0, T_1.
Consider randomly generated string $S = s_1, s_2, \ldots, \in \mathcal{X}^*$.

The tree used to encode s_i is modelled by a two state ergodic Markov Chain.

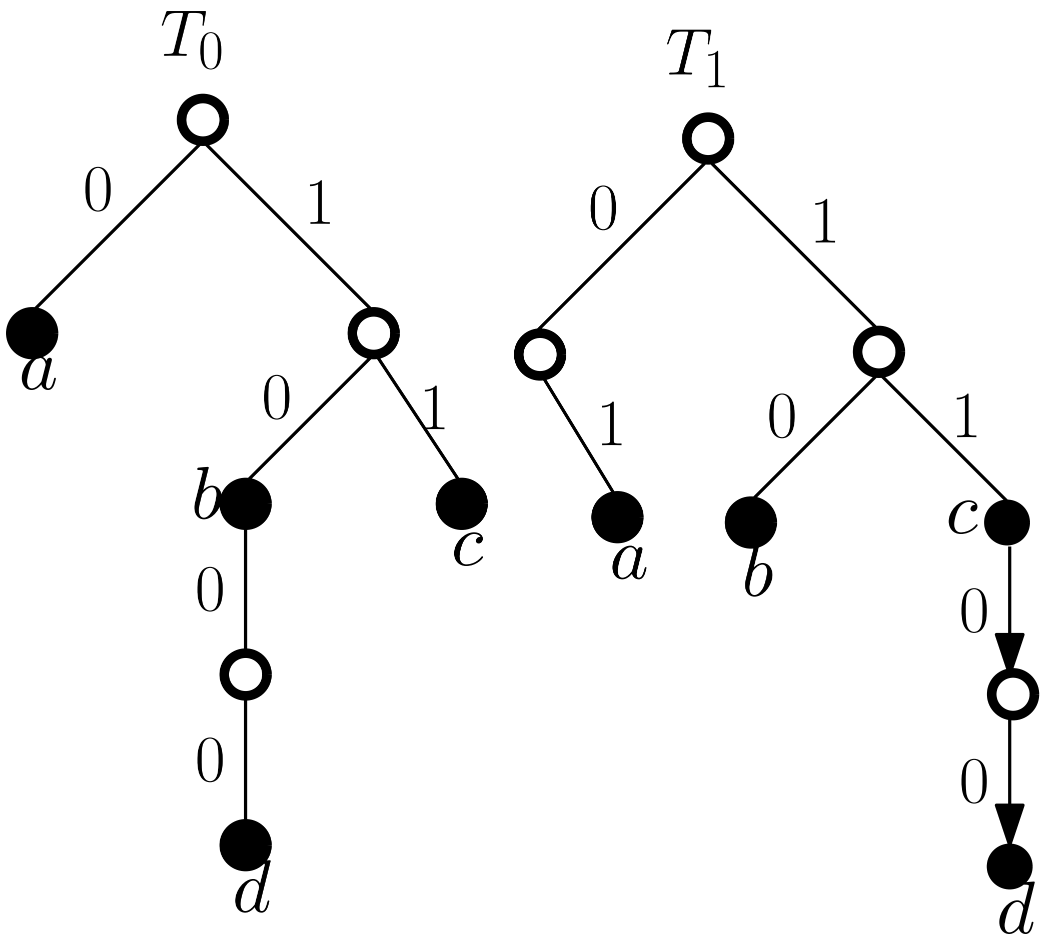
> Problem: Find T_0, T_1 that minimize $L_{AIFV}(T_0, T_1)$

$$L_{AIFV}(T_0, T_1) = P(0|T_0, T_1)L(T_0) + P(1|T_0, T_1)L(T_1)$$

stat. prob of being in T_0 — cost of T_0 — stat. prob of being in T_1 — cost of T_1

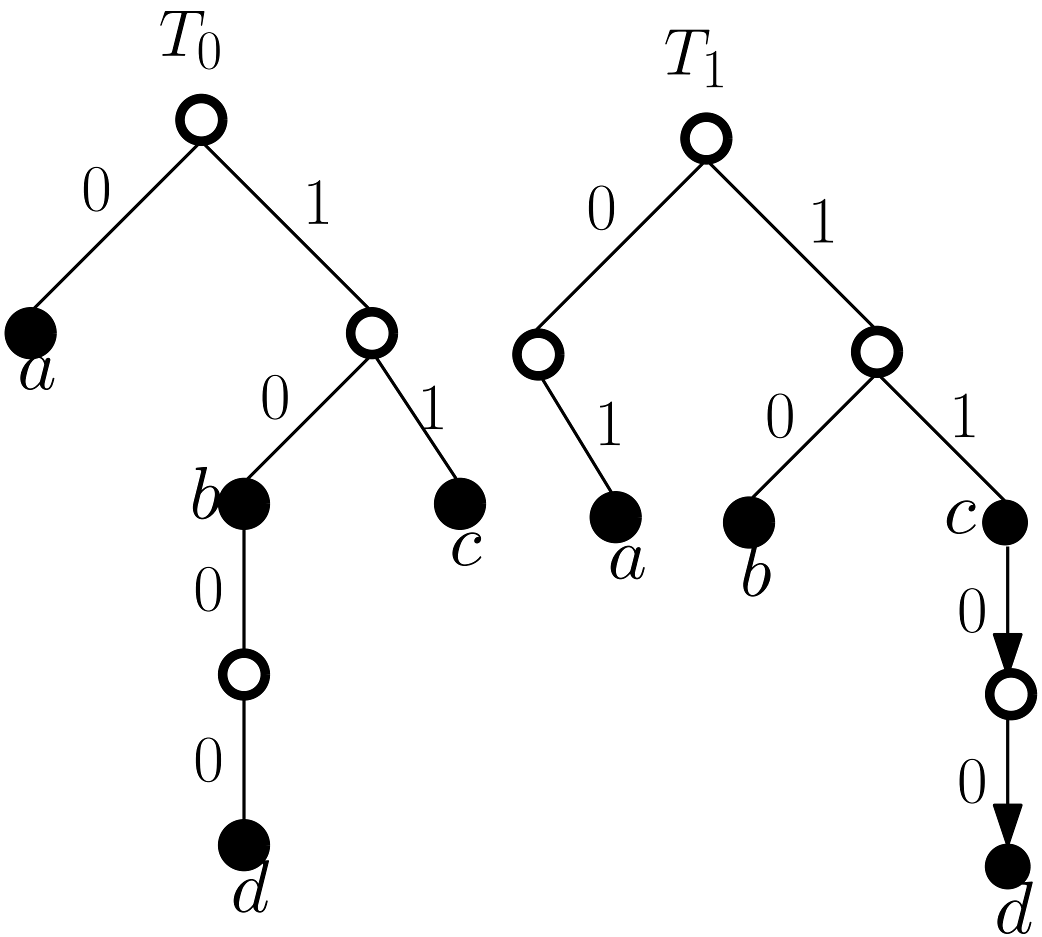


T_0
0
1
a
0
1
b
c
0
0
d
T_1
0
1
1
0
1
a
b
c
0
0
d



$p_X(a) = 0.5 \quad p_X(b) = 0.25$

$p_X(c) = 0.2 \quad p_X(d) = 0.05$



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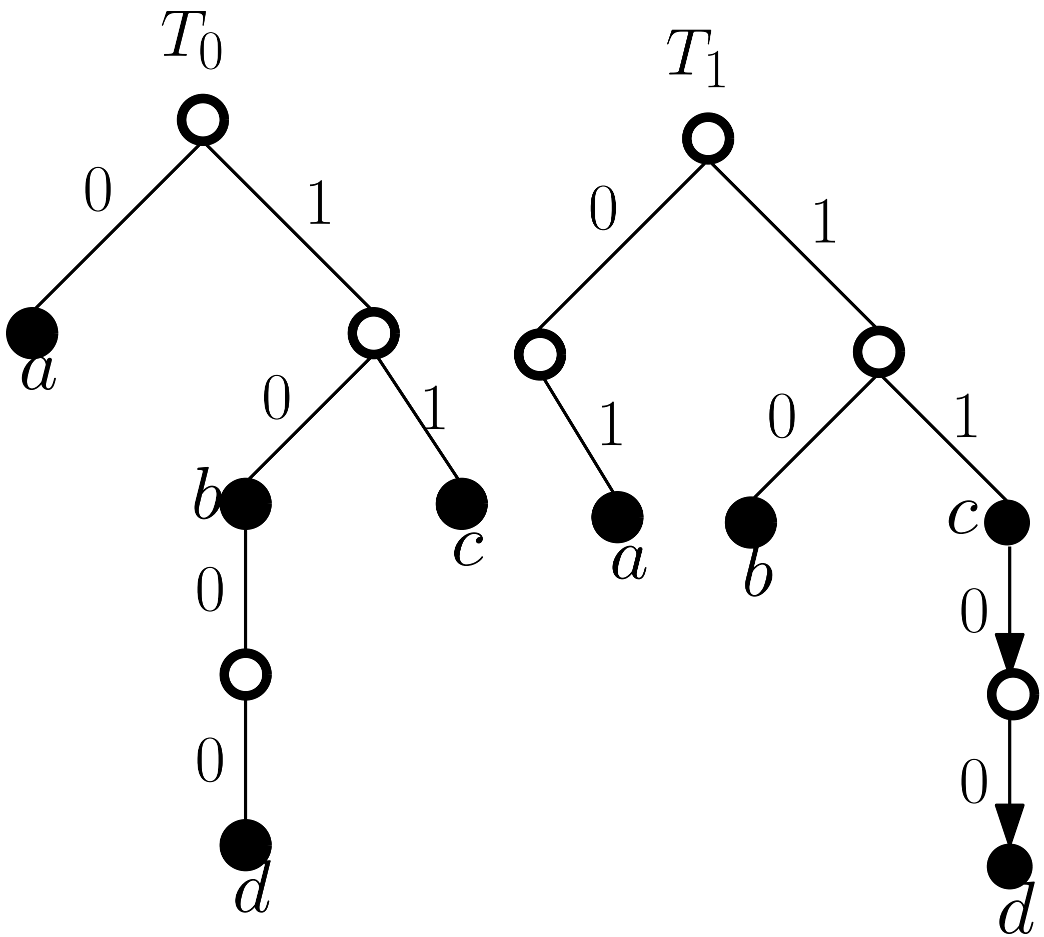

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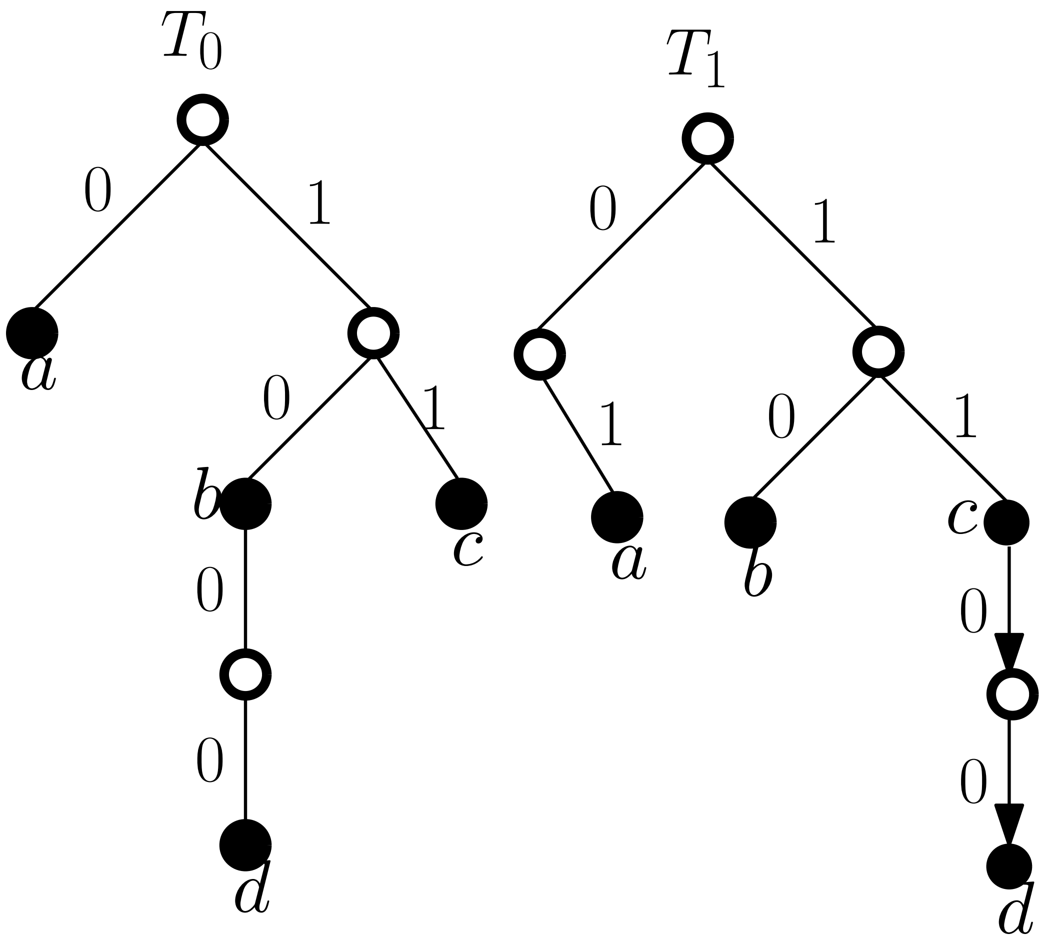

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AIFV-2 Construction Algorithm

- Yamamoto et al. proved that this Algorithm constructs optimal AIFV-2 Codes.

Algorithm [Yamamoto et al]

$m \leftarrow 0$

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They proved that Algorithm terminates after finite number of iterations, but no bound on number of iterations was known.

Outline

- Introduction
- AIFV-2 codes: cost and algorithm
- A Geometric Interpretation of the old algorithm
    - A New Binary Search Algorithm
    - An Ellipsoid Algorithm
- Extensions to AIFV-k codes (skip)
- Summing up and open questions

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For fixed T_0, T_1, these look like eqns of a line.

Eqn for x-coord of intersection of the 2 lines

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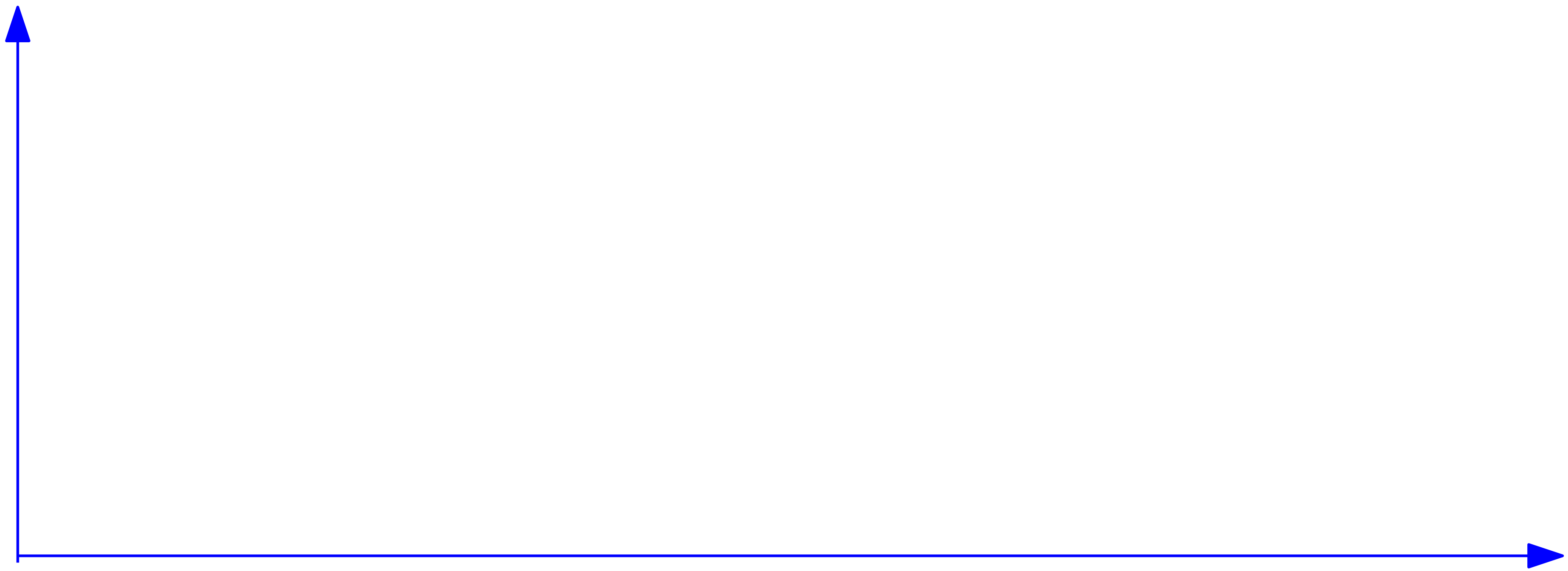

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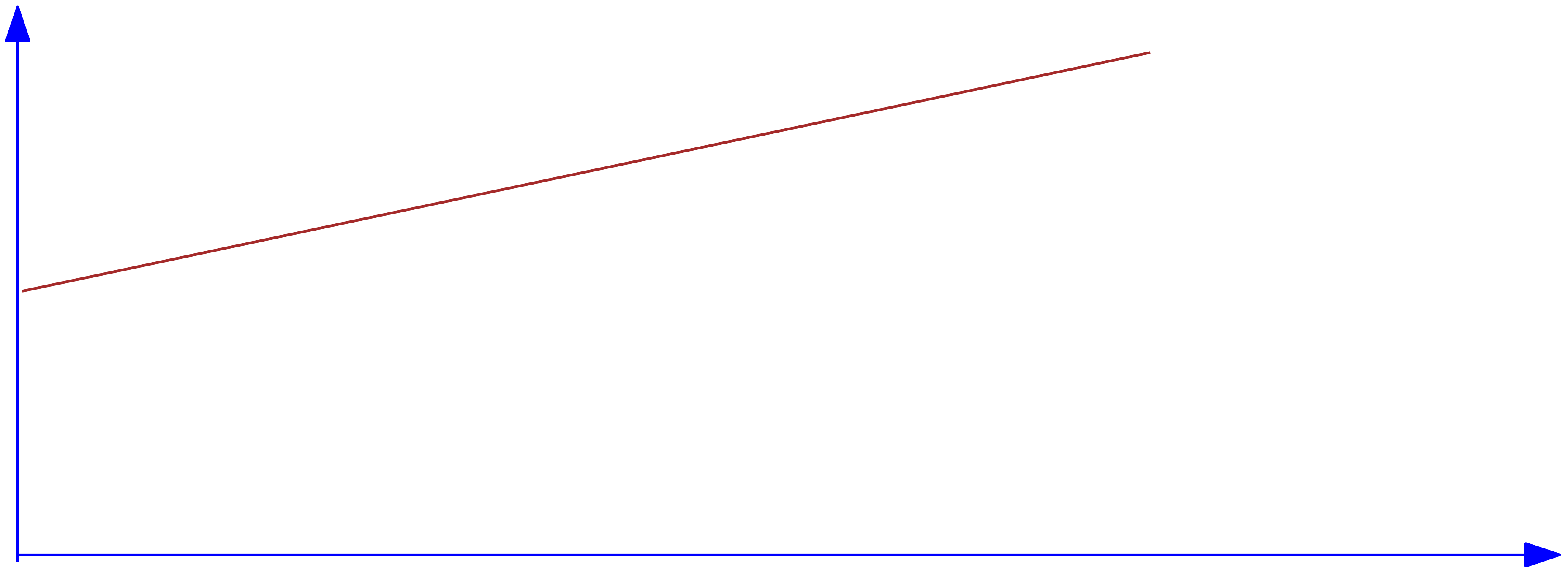

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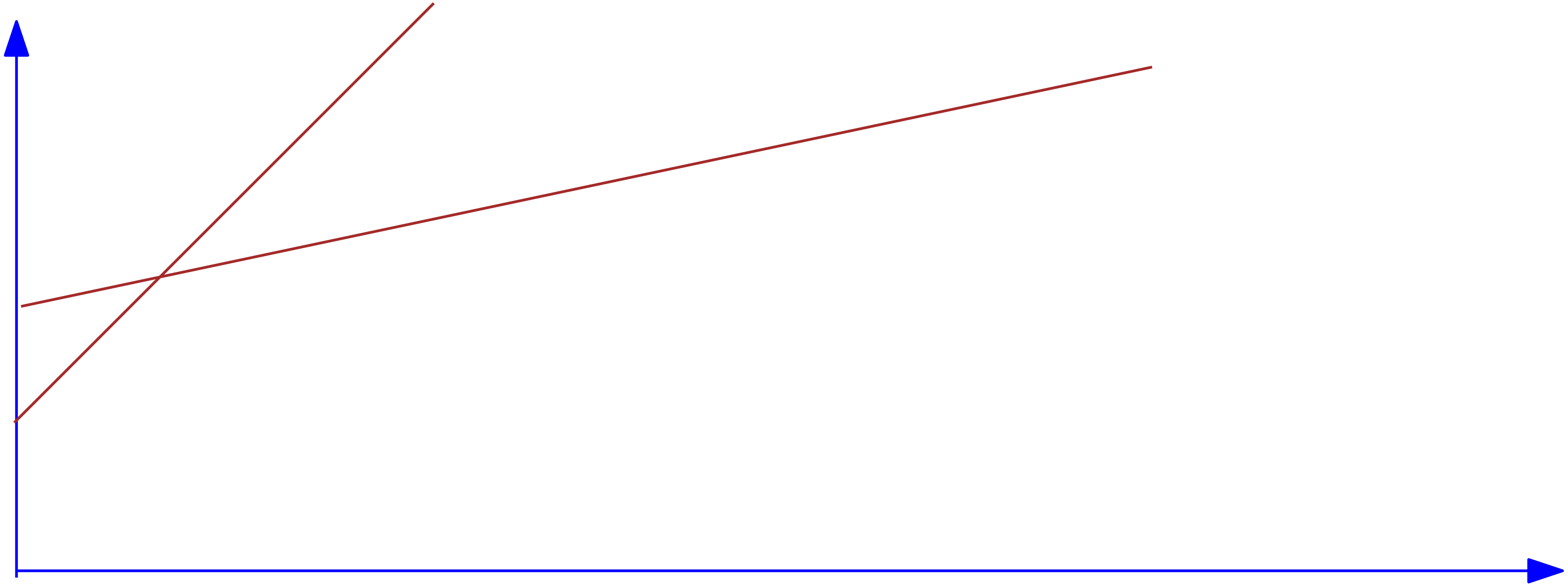

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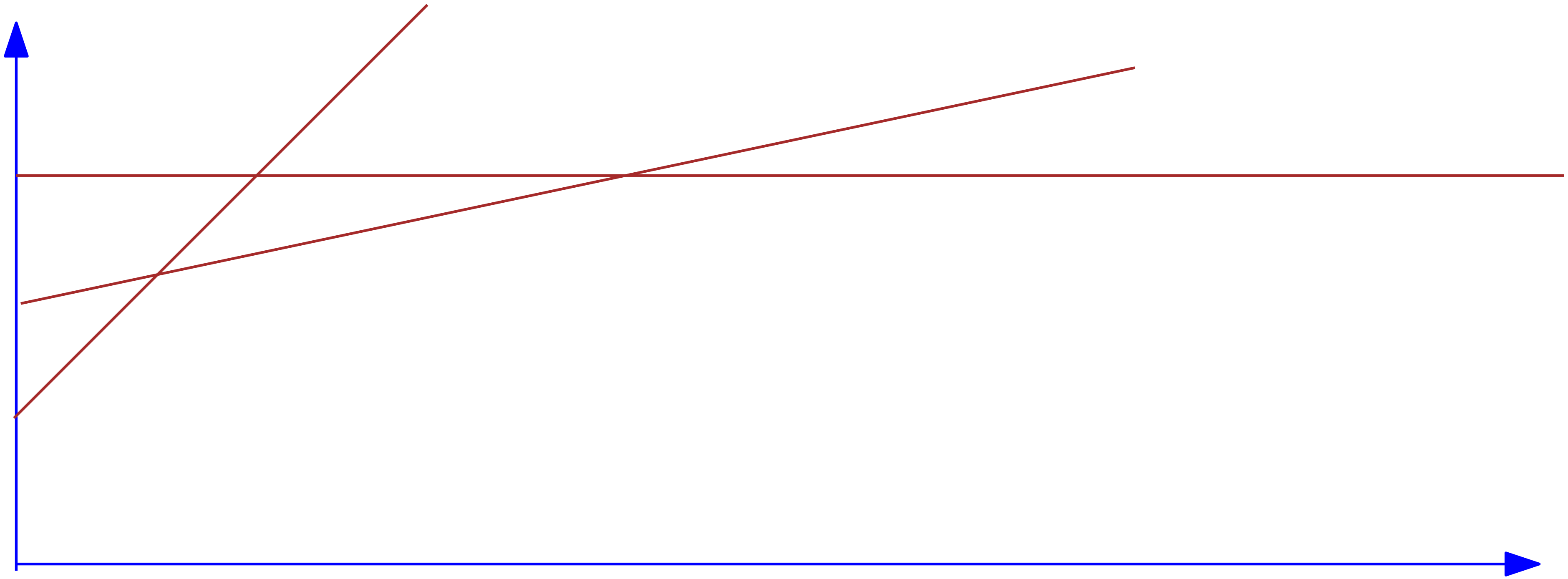

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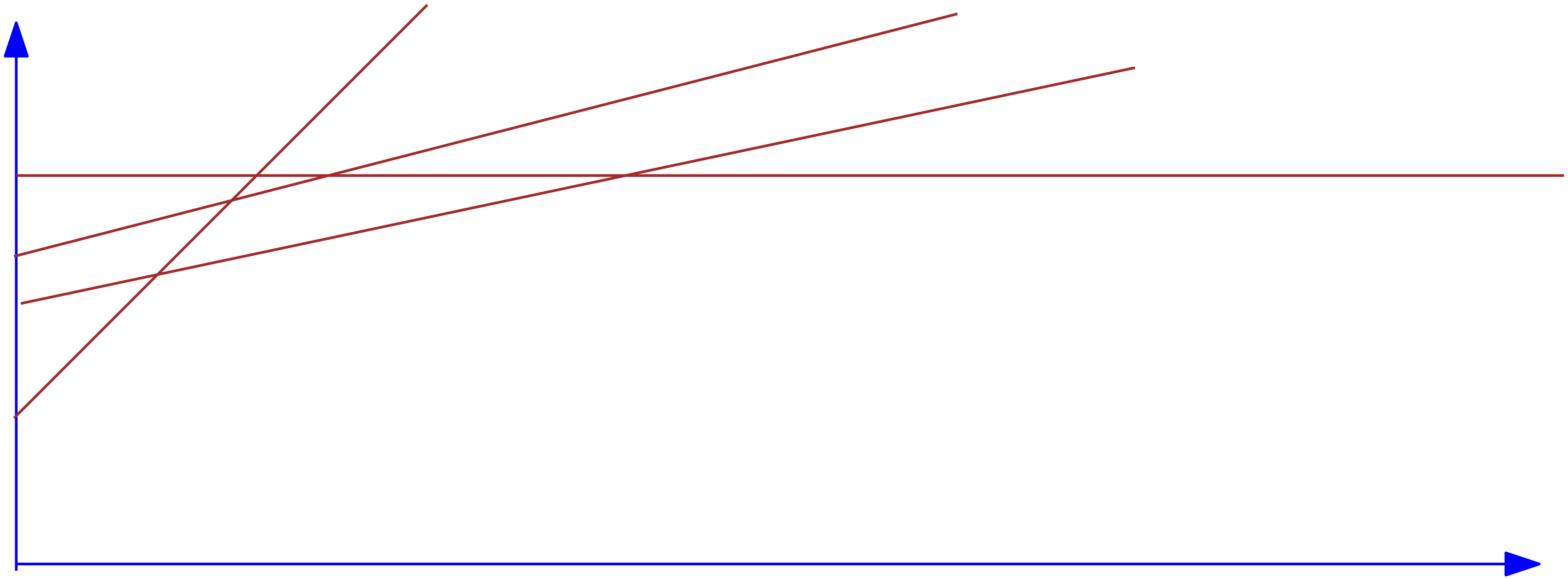

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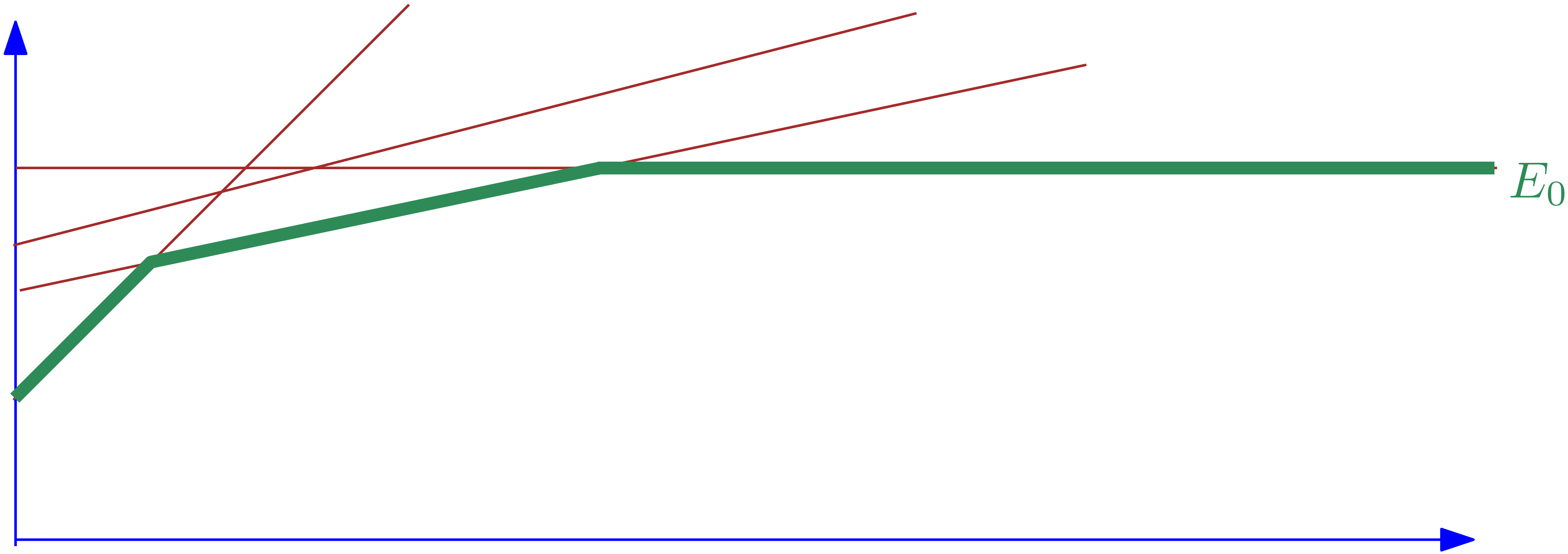

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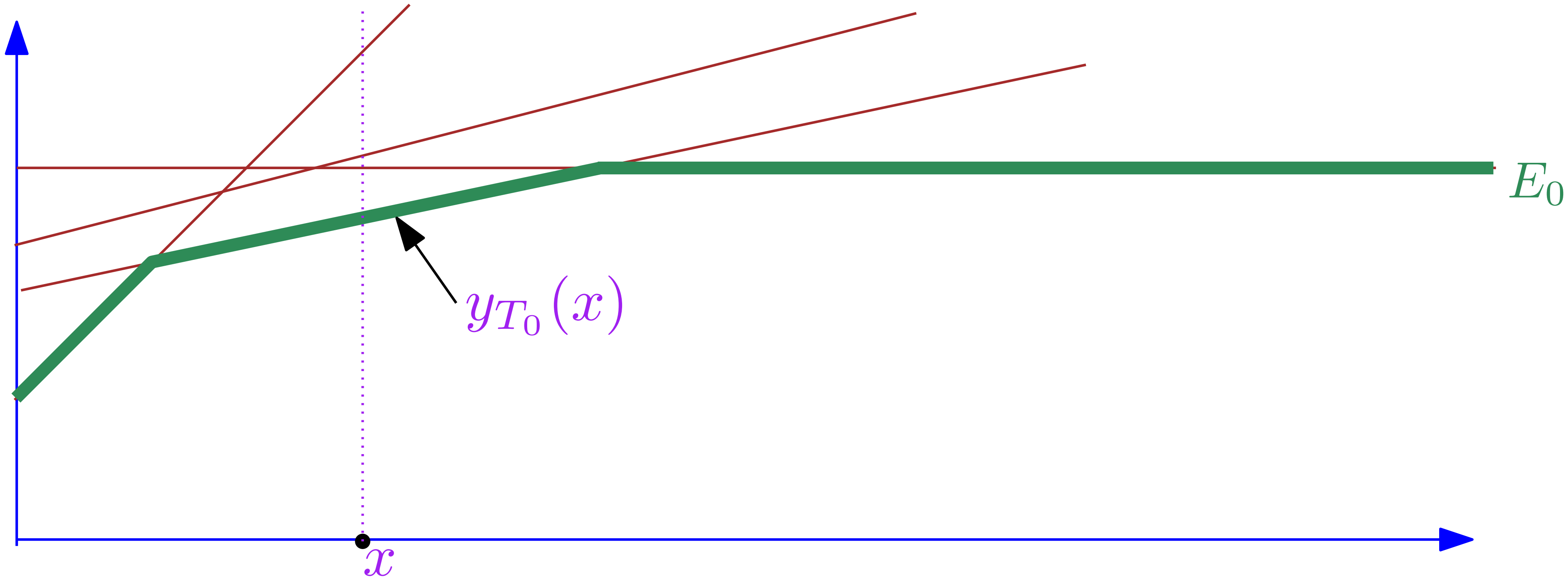

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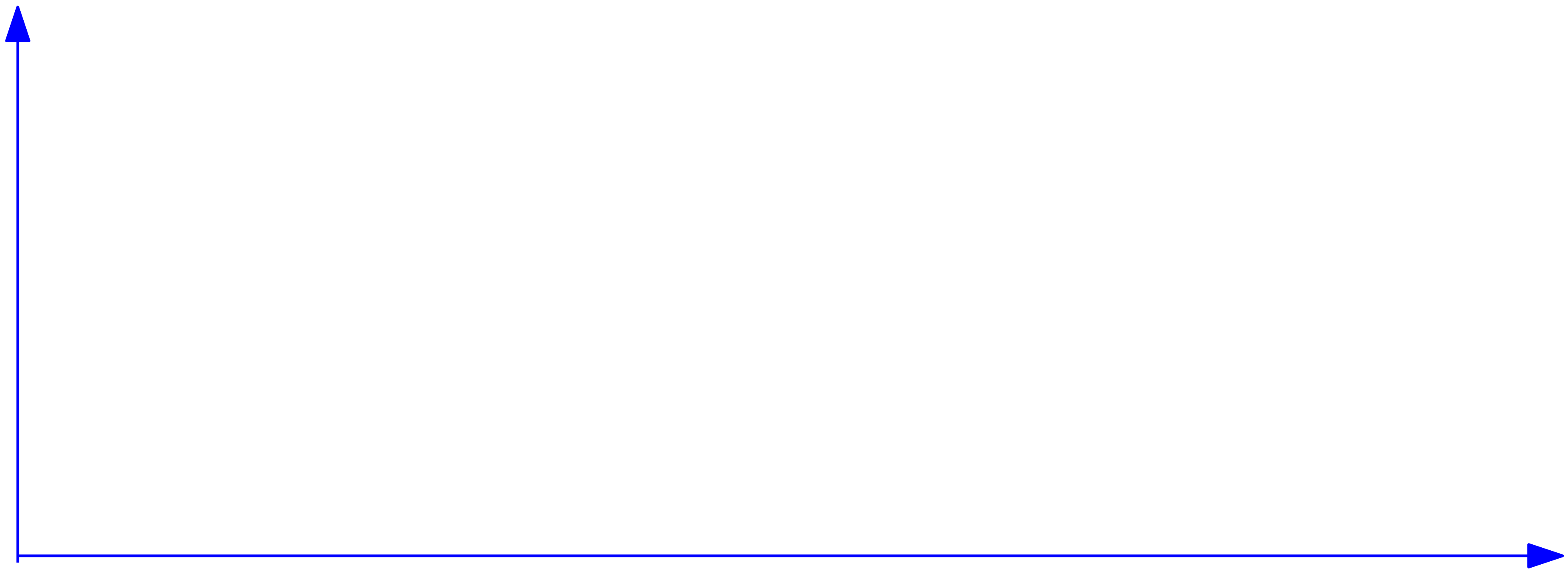

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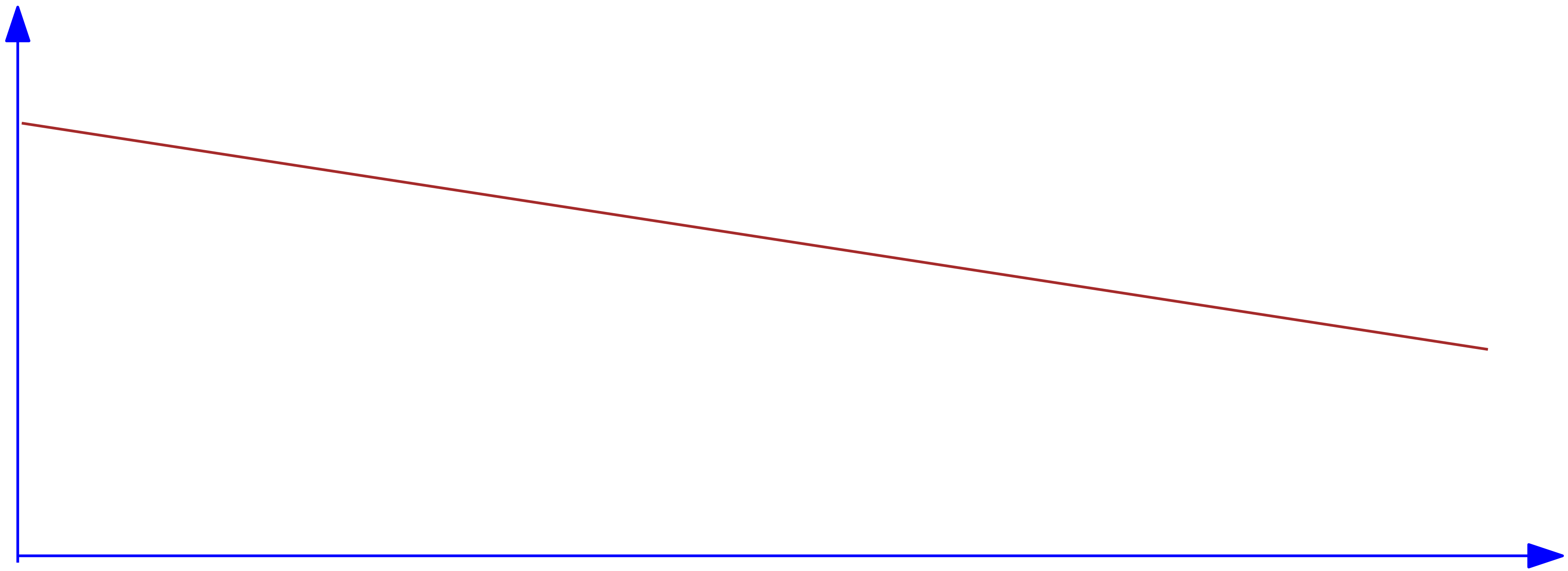

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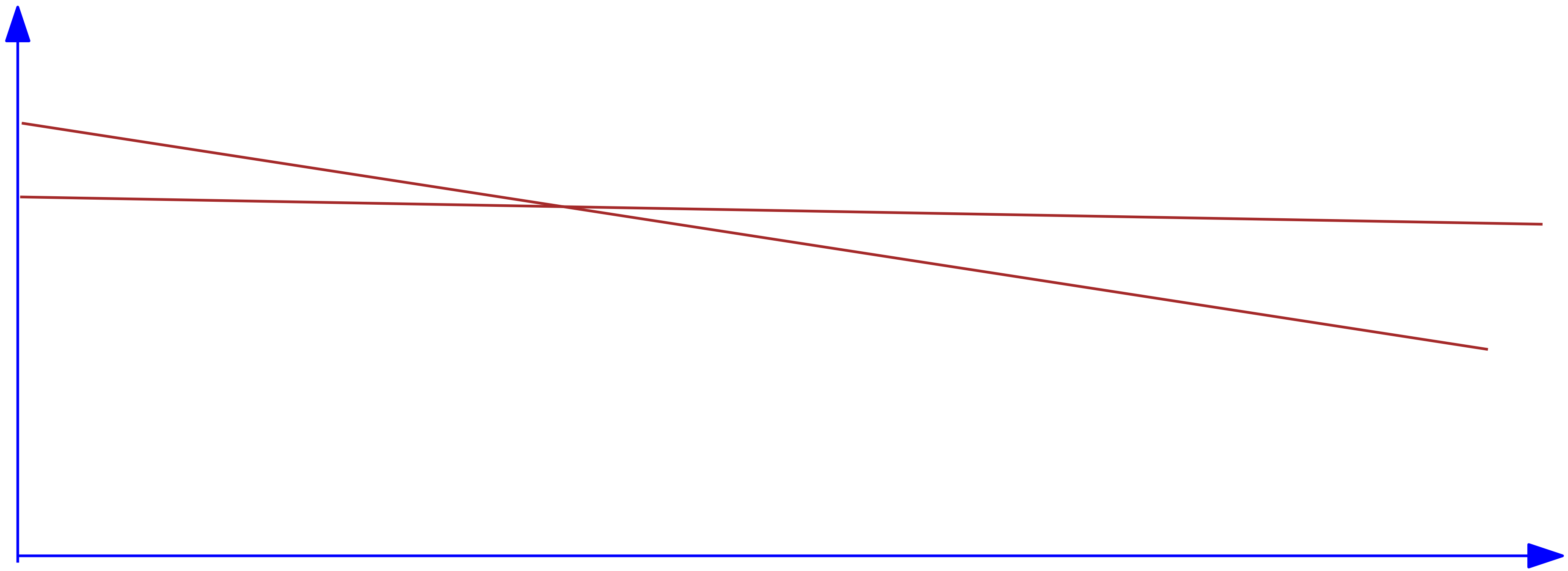

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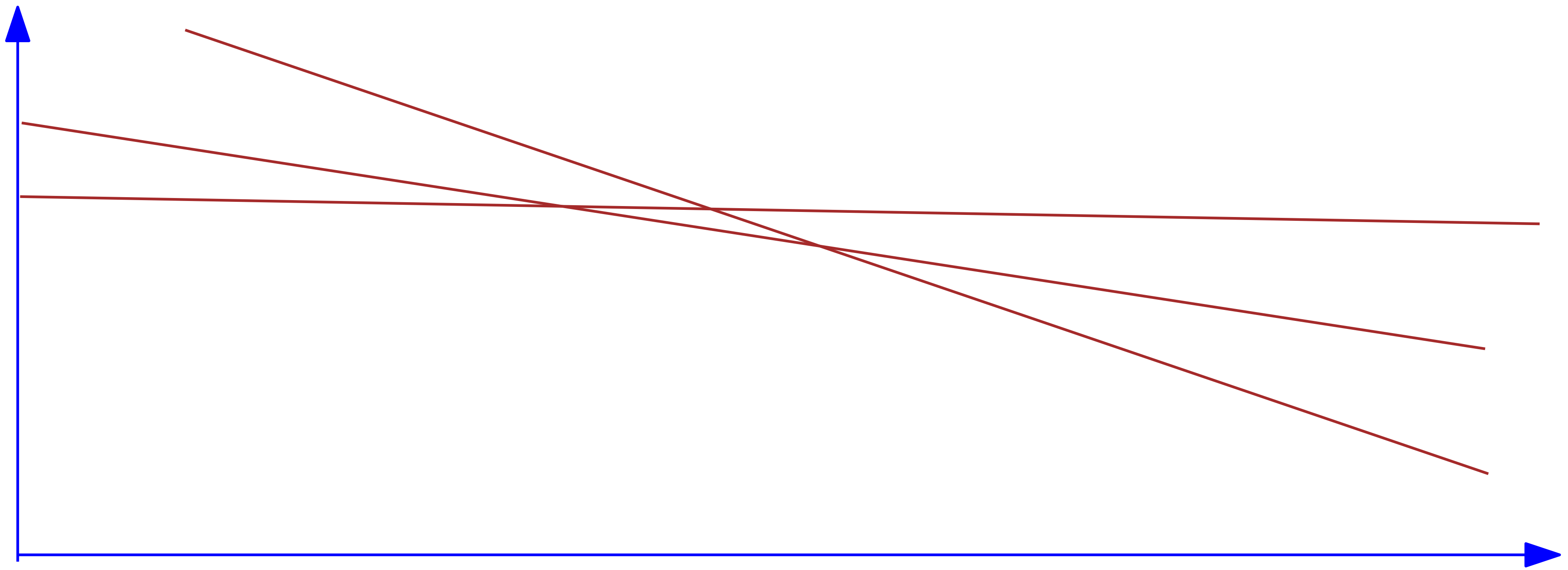

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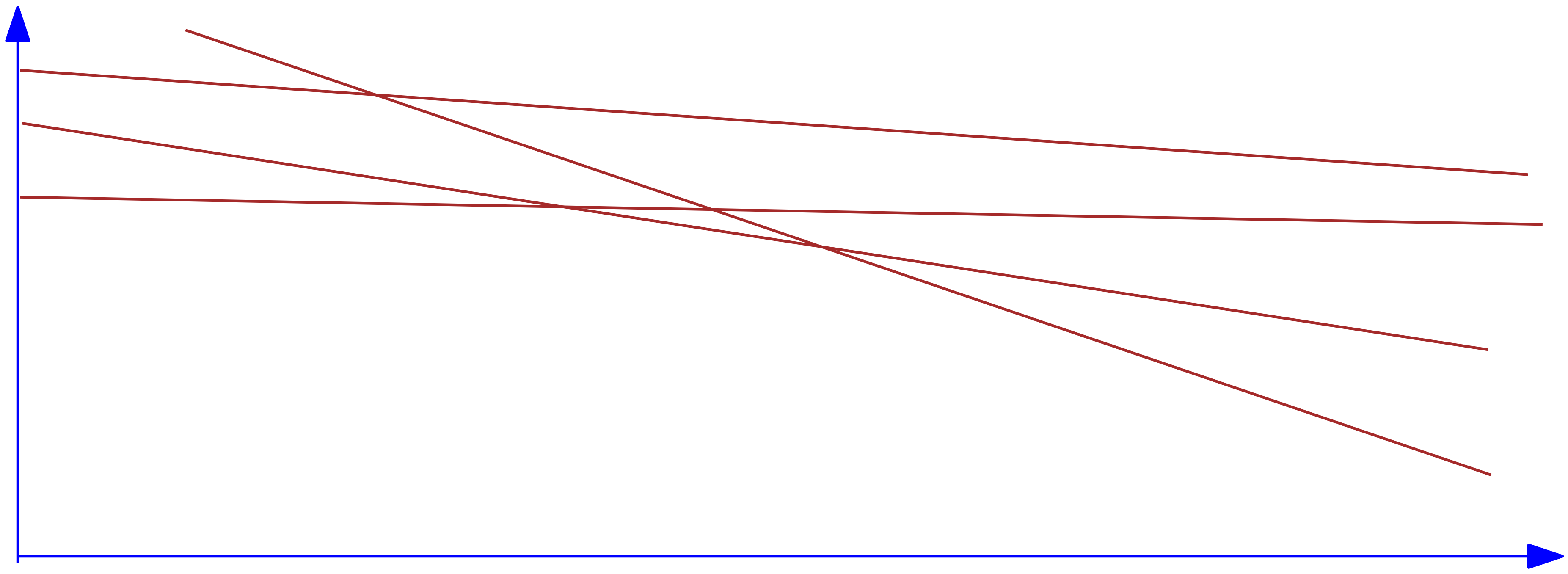

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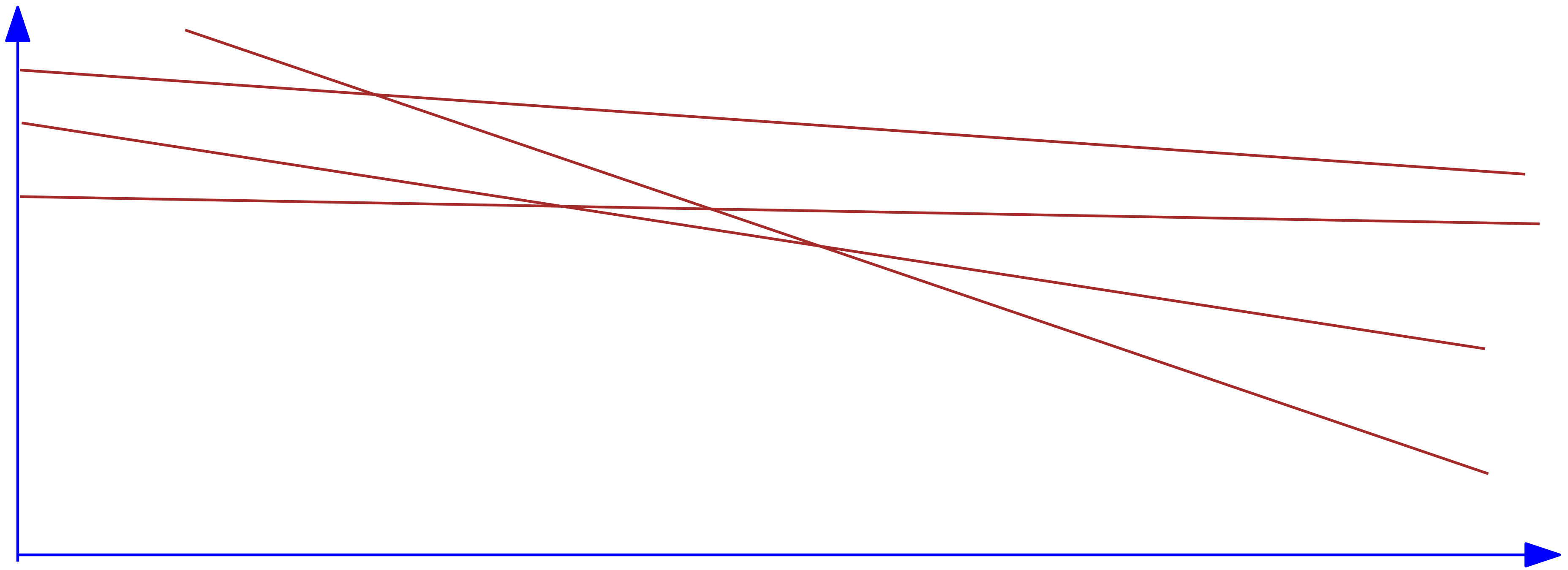

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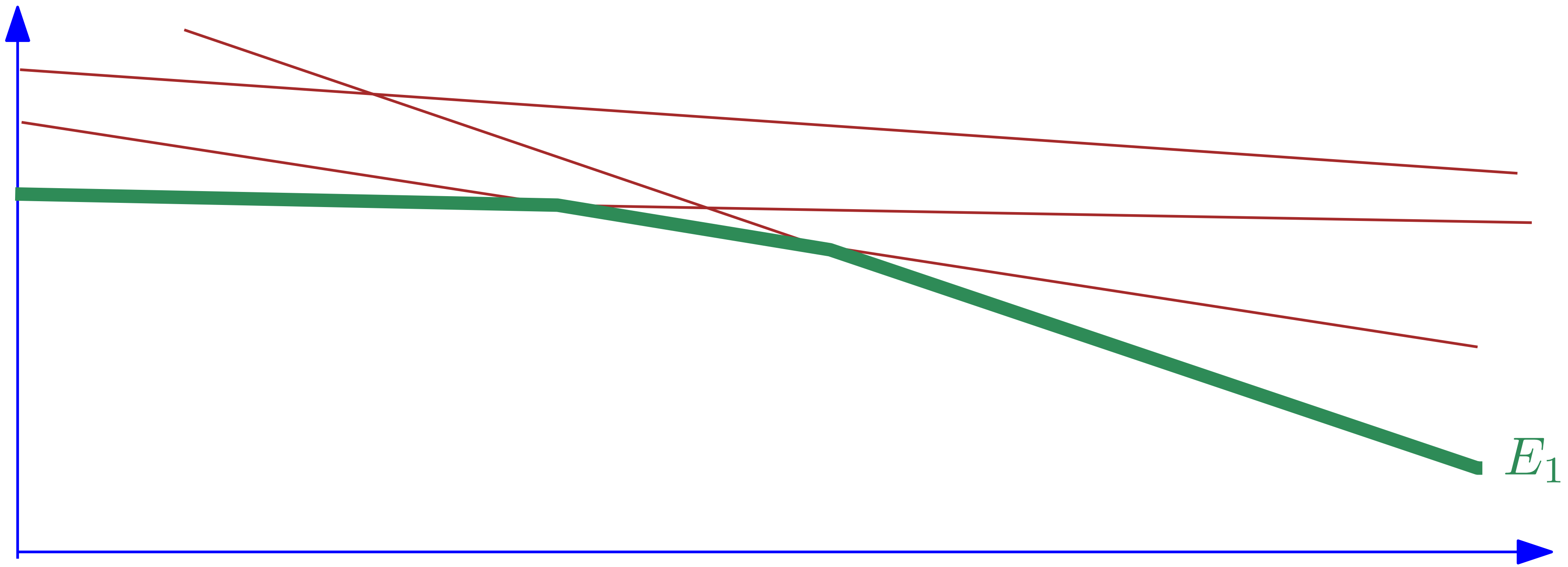

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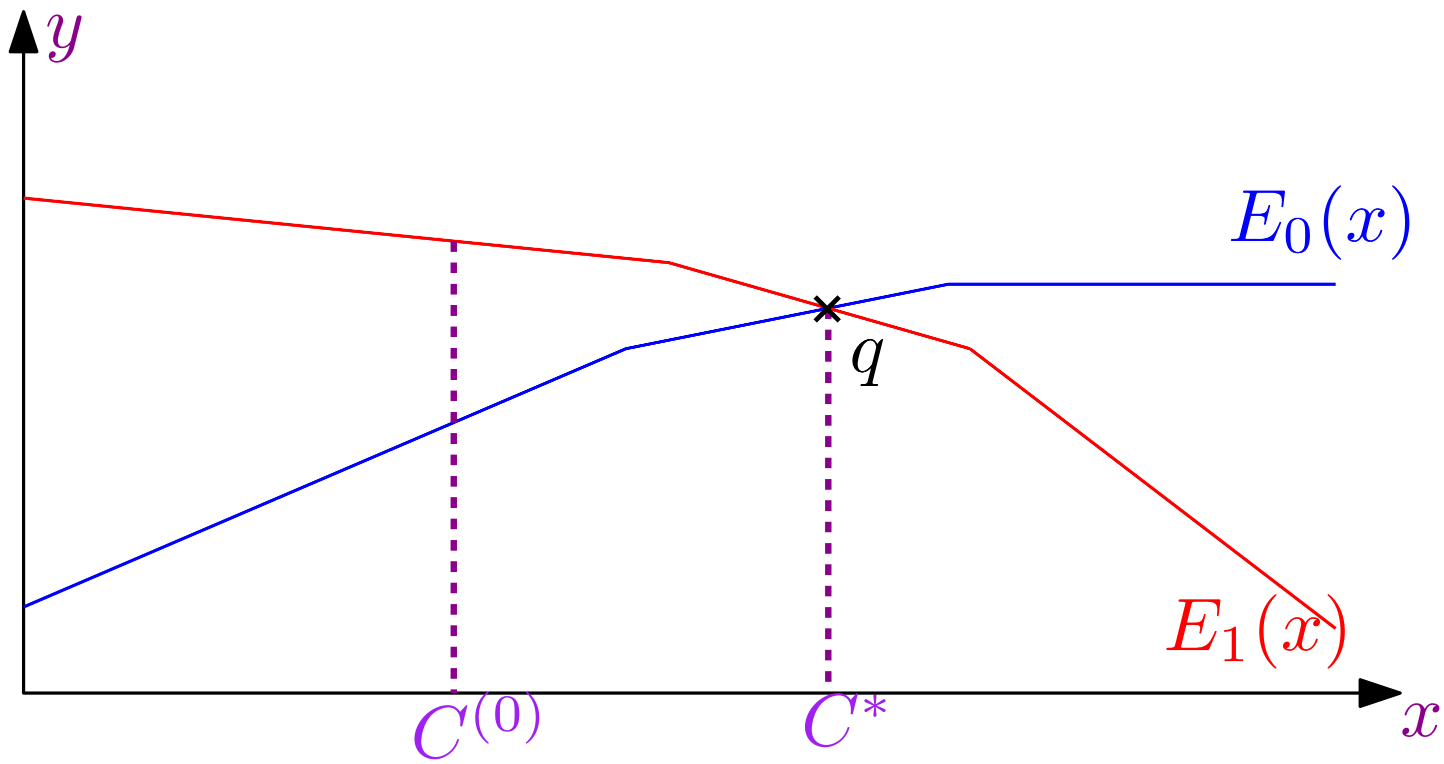
- Because $E_0(x)$ has positive slope and $E_1(x)$ negative slope they intersect at a unique point q with x-coordinate $x = C^*$.



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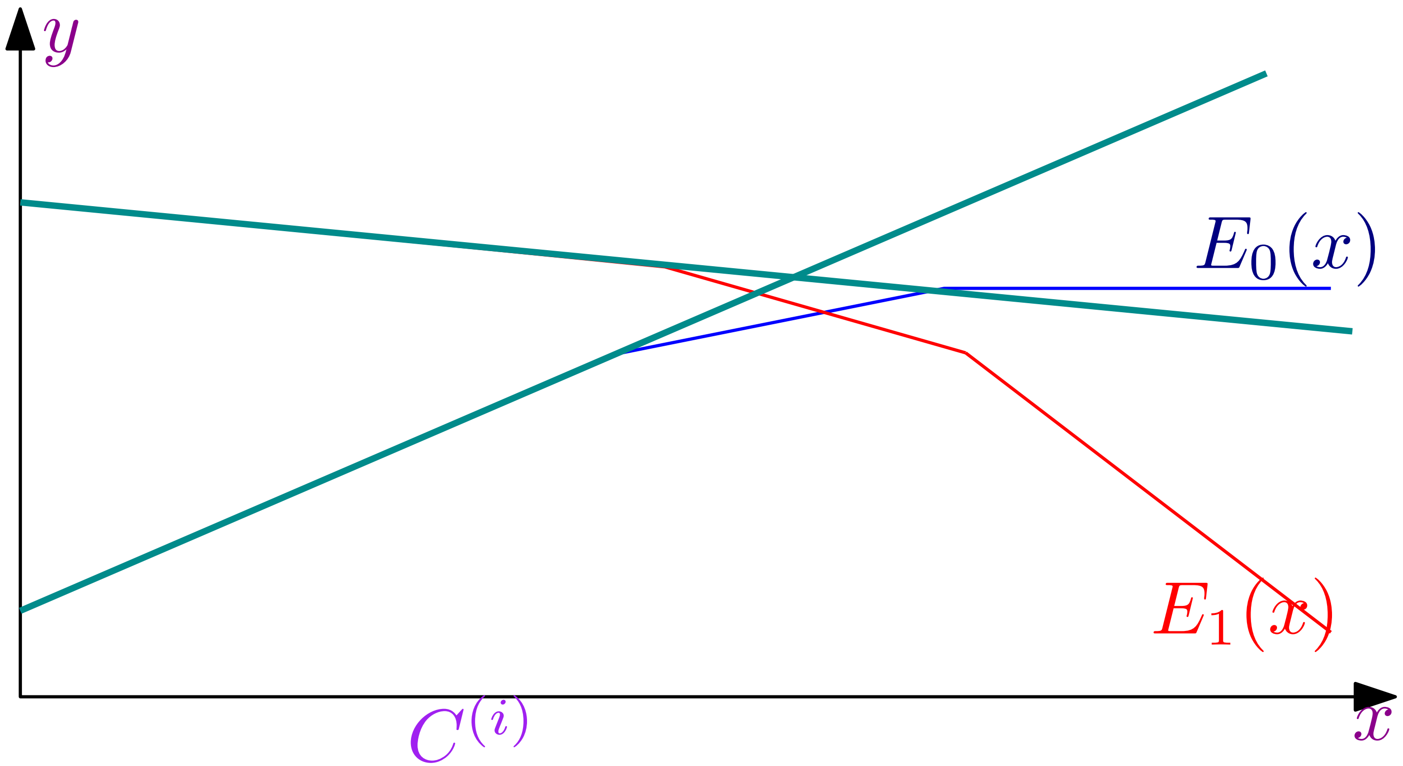

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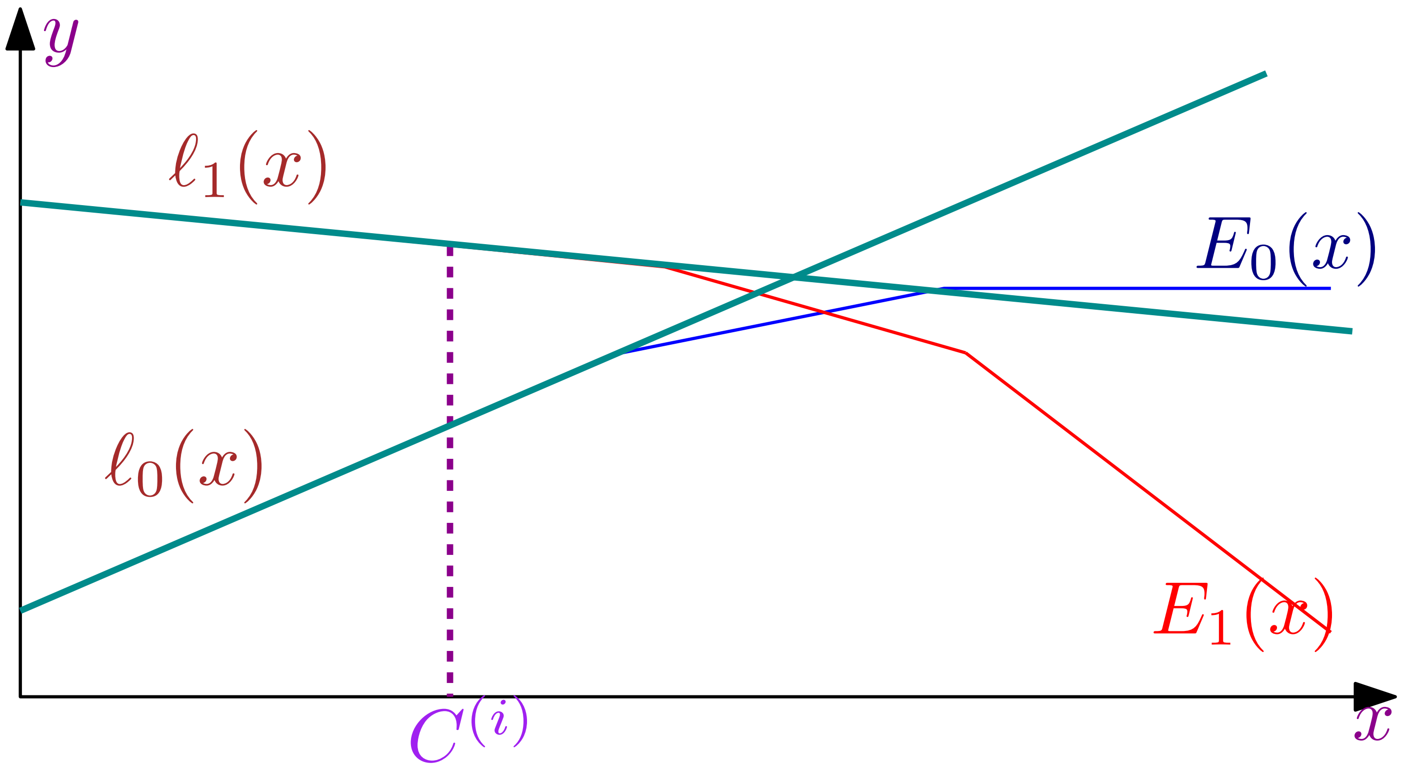

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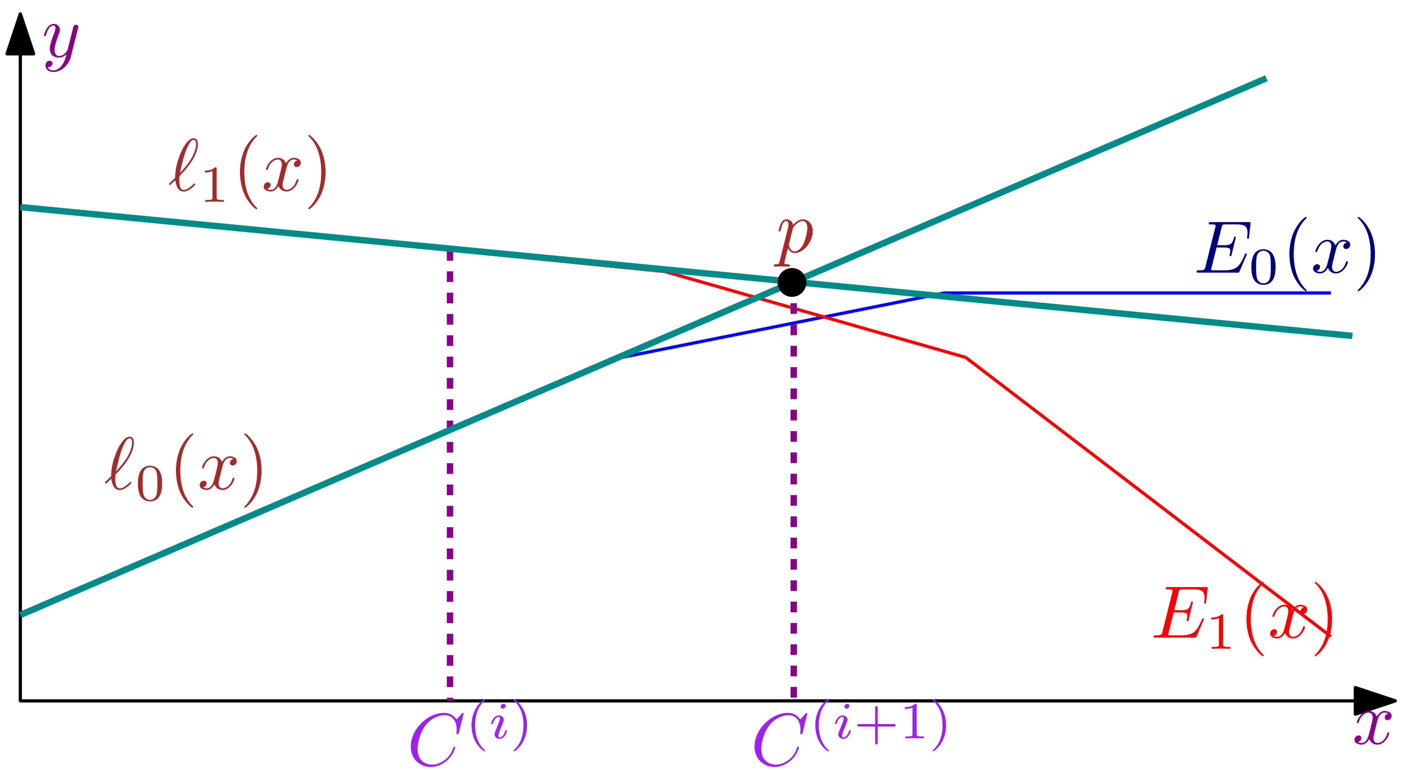

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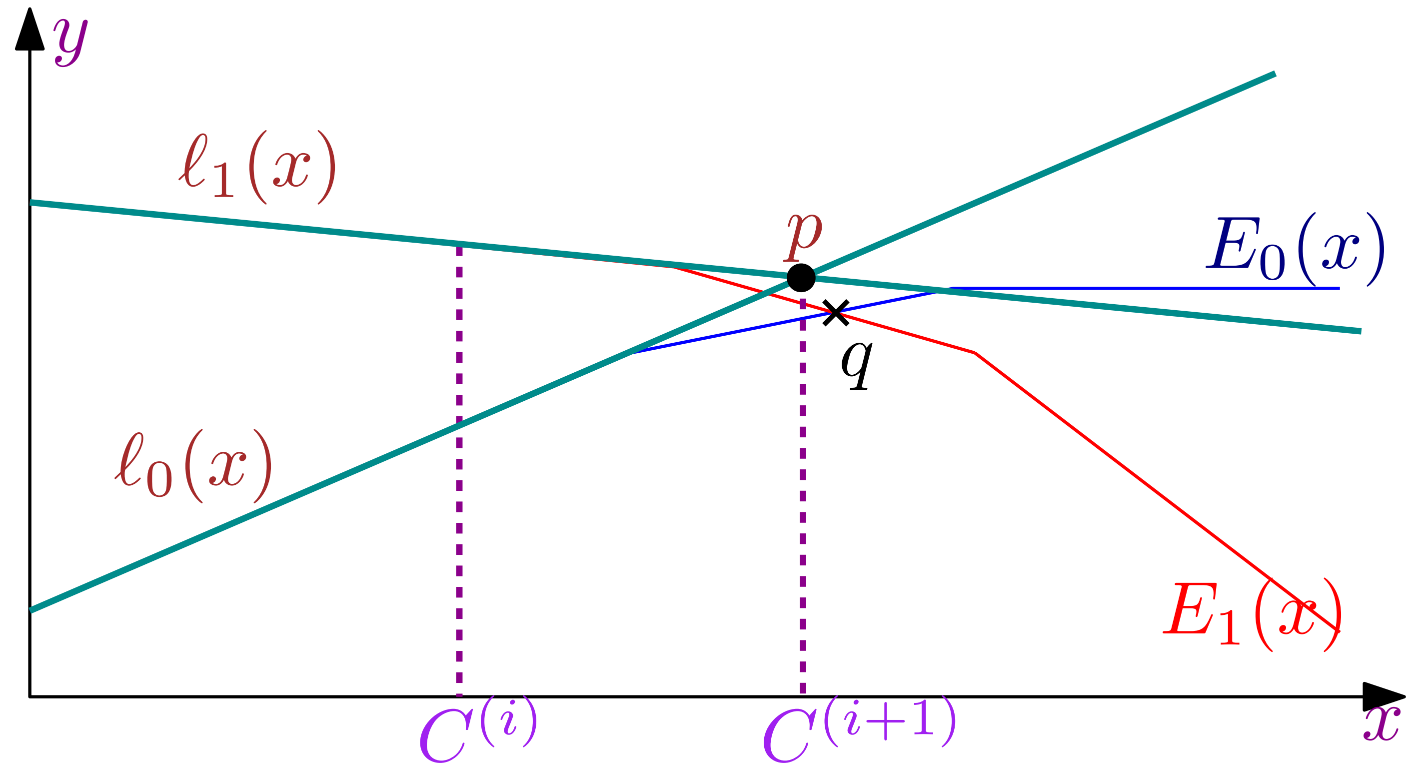
Unless $p = q$, the unique intersection of $E_0(x)$ and $E_1(x)$, this process will continue, so it can only terminate if $C^{(i+1)} = C^*$.



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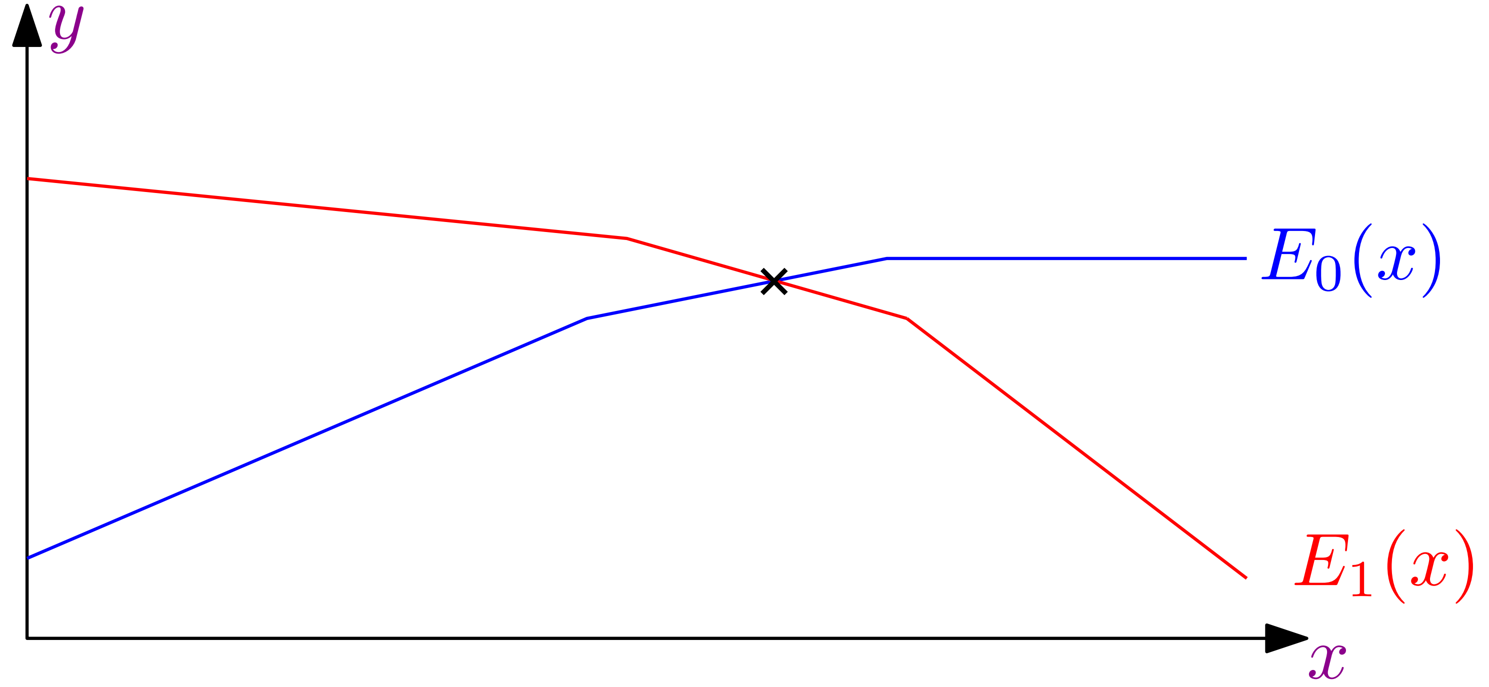

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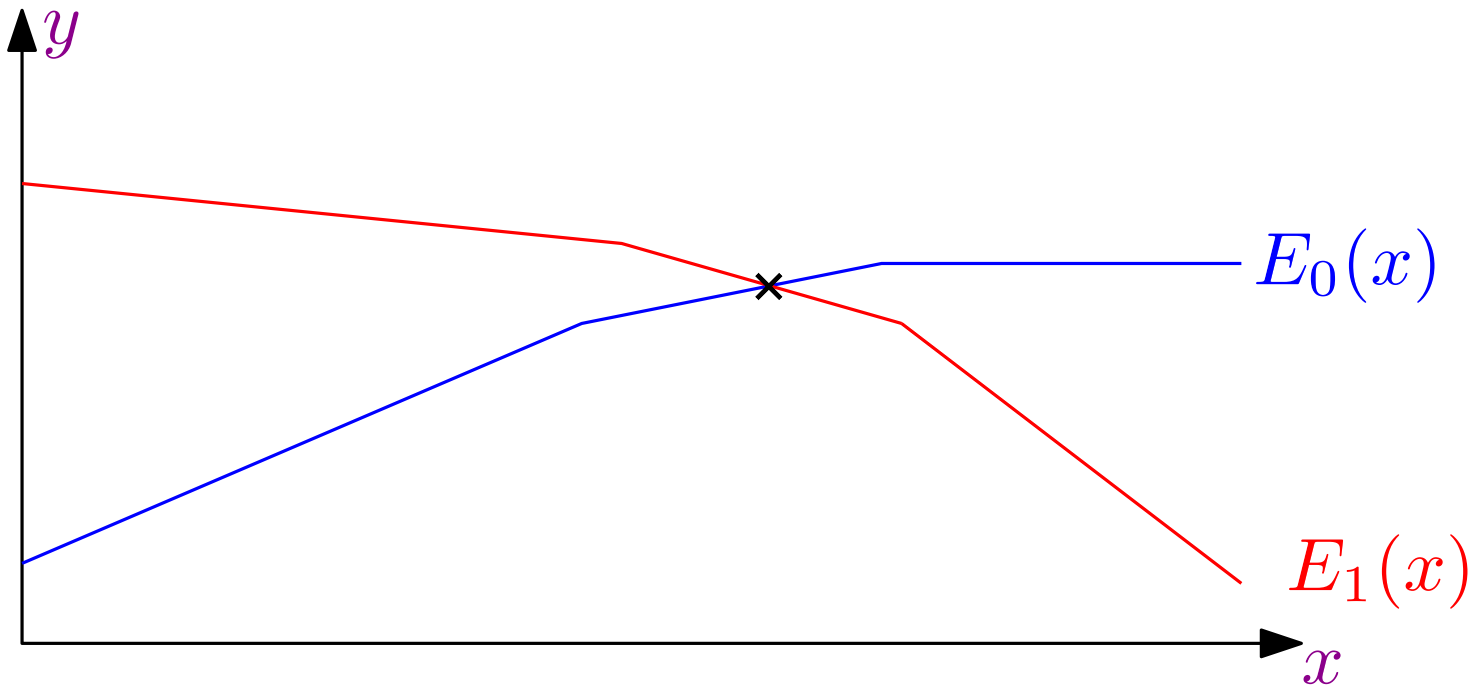

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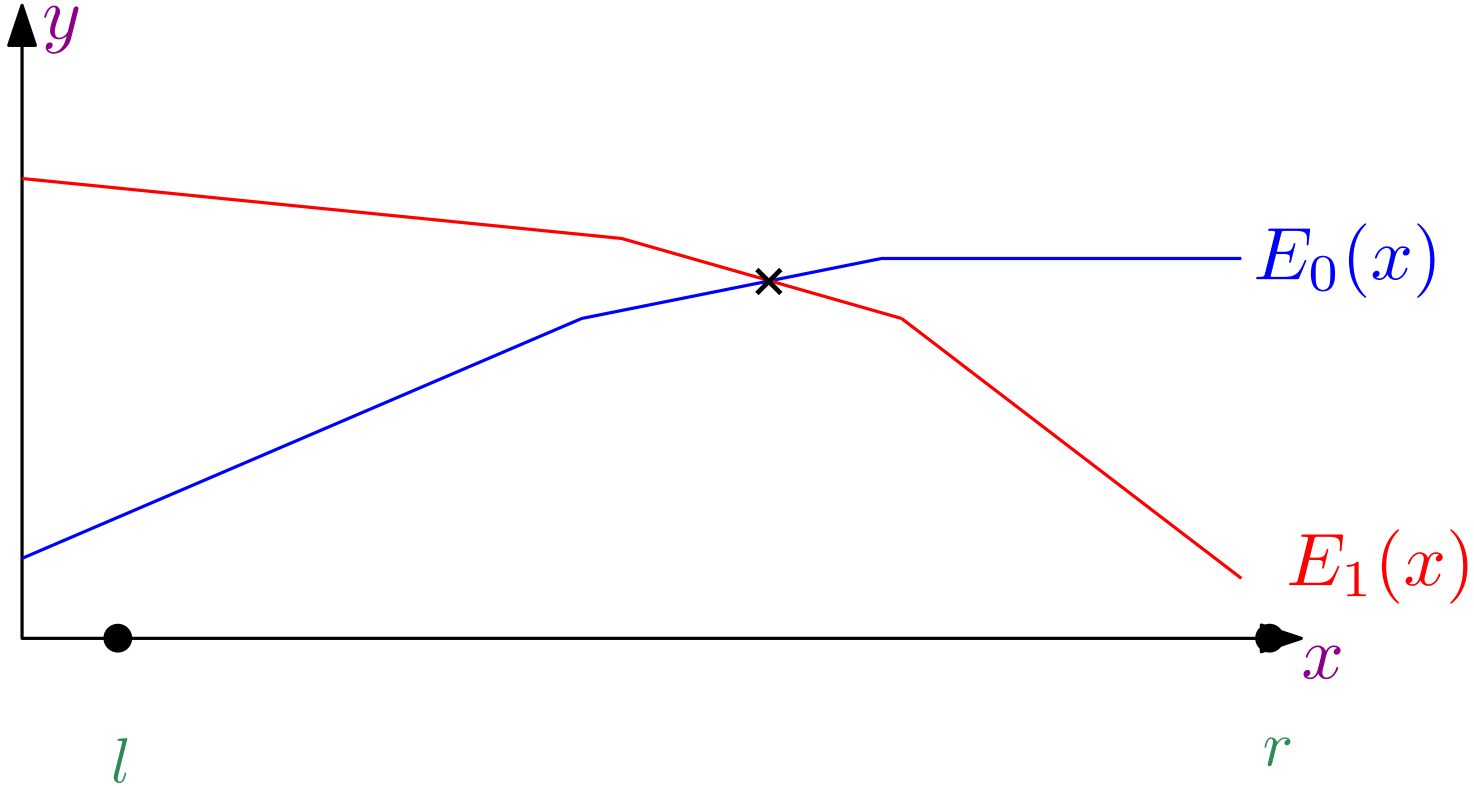

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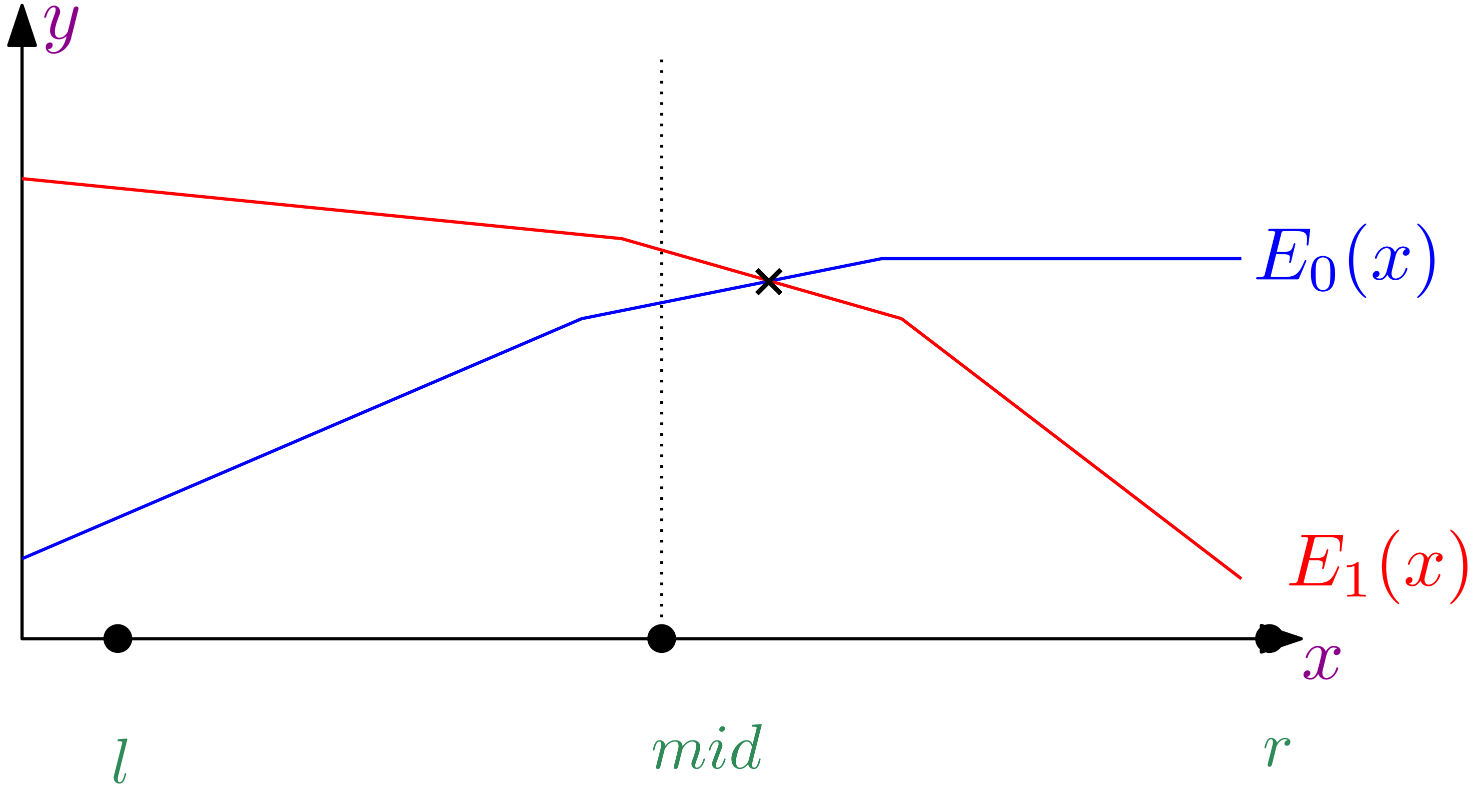

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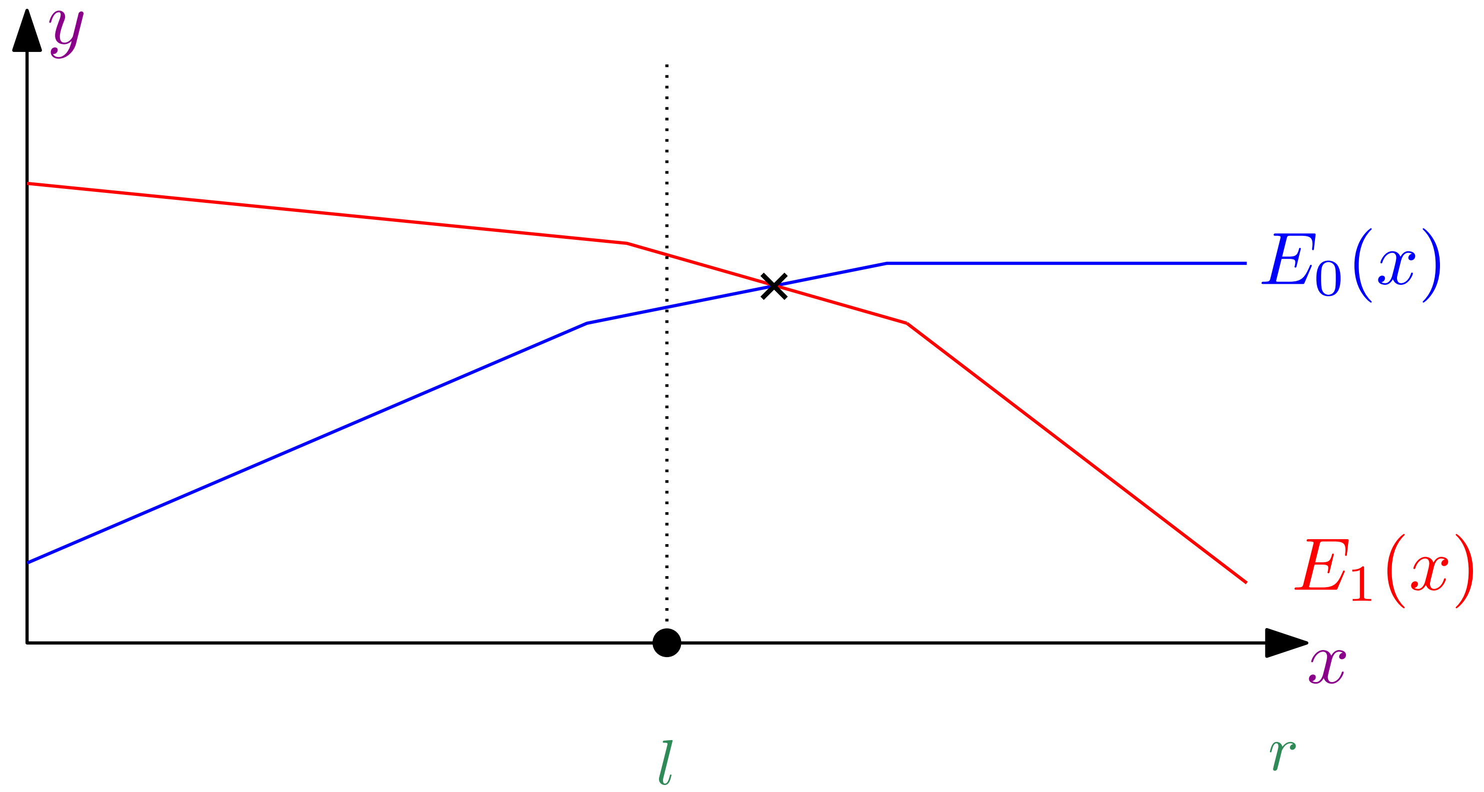

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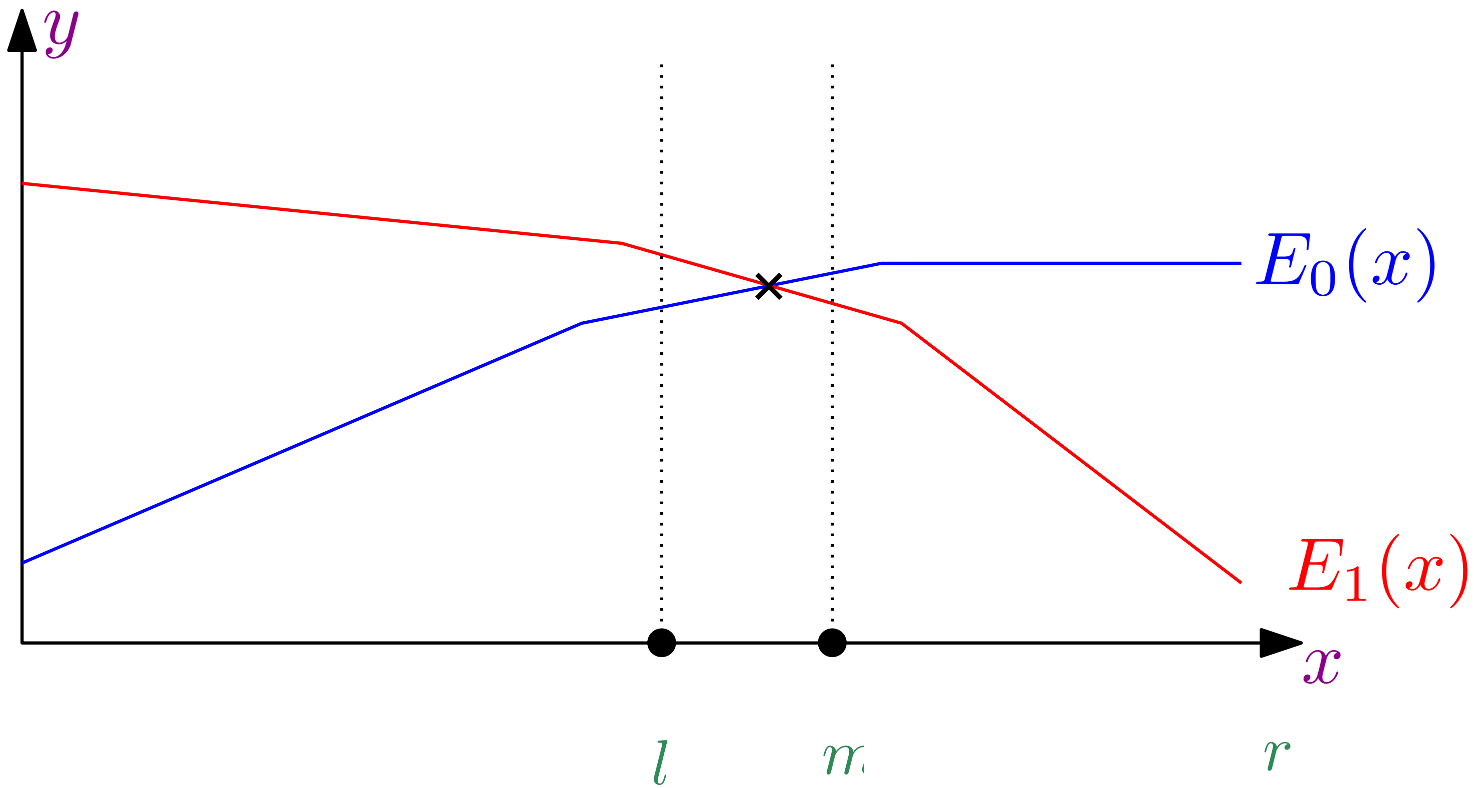

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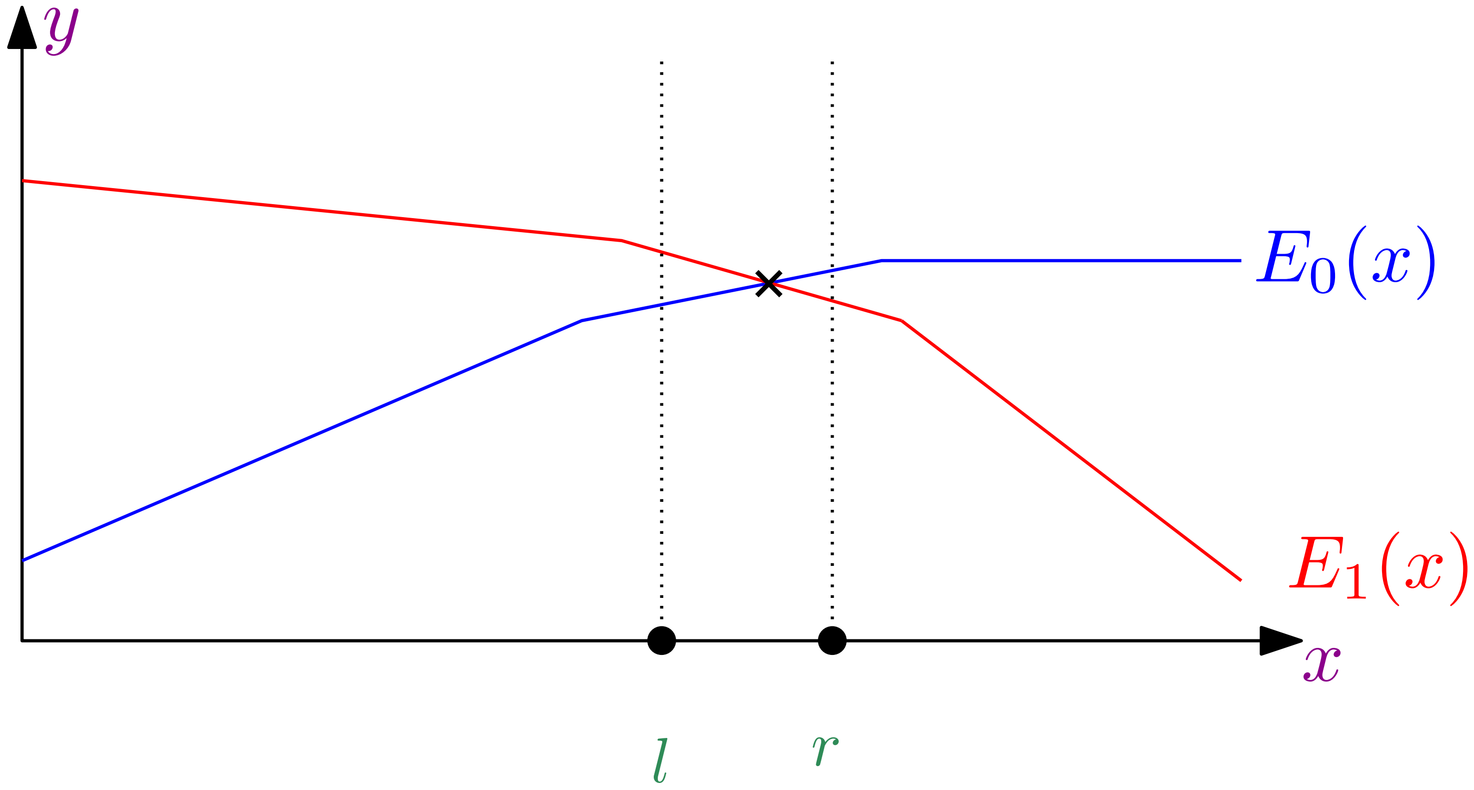

- Observation, $C^* \in [0, 1]$ and
  $C^* \in [l, r] \quad \Leftrightarrow \quad E_0(l) < E_1(l)$ and $E_1(r) < E_0(r)$

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- After $O(\log(\frac{1}{2^{-2b}})) = O(b)$ queries, binary search can terminate.

- In each query , the algorithm uses $O(n^5)$ time dynamic programming to find the trees (lines) on the lower envelopes for current value of C.

- Algorithm takes $O(n^5 b)$ time.
  This is first (weakly) polynomial algorithm for constructing AIFV-2 Codes.

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- Ellipsoid Method: Let $K \in \mathbb{R}^m$ be a closed convex set and $c \in \mathbb{Q}^m$. Assume that we have a separation oracle for K. Also assume we know positive numbers R and ϵ such that $K \subset B(0, R)$ and $Vol(K) > \epsilon$. Then with the ellipsoid method, in time polynomial in $m, \log \epsilon, \log R,$ and $\log \Delta$, we get a solution $x_0 \in K$ such that

$$c^T x_0 \geq \max\{c^T x | x \in K\} - \Delta |c|$$

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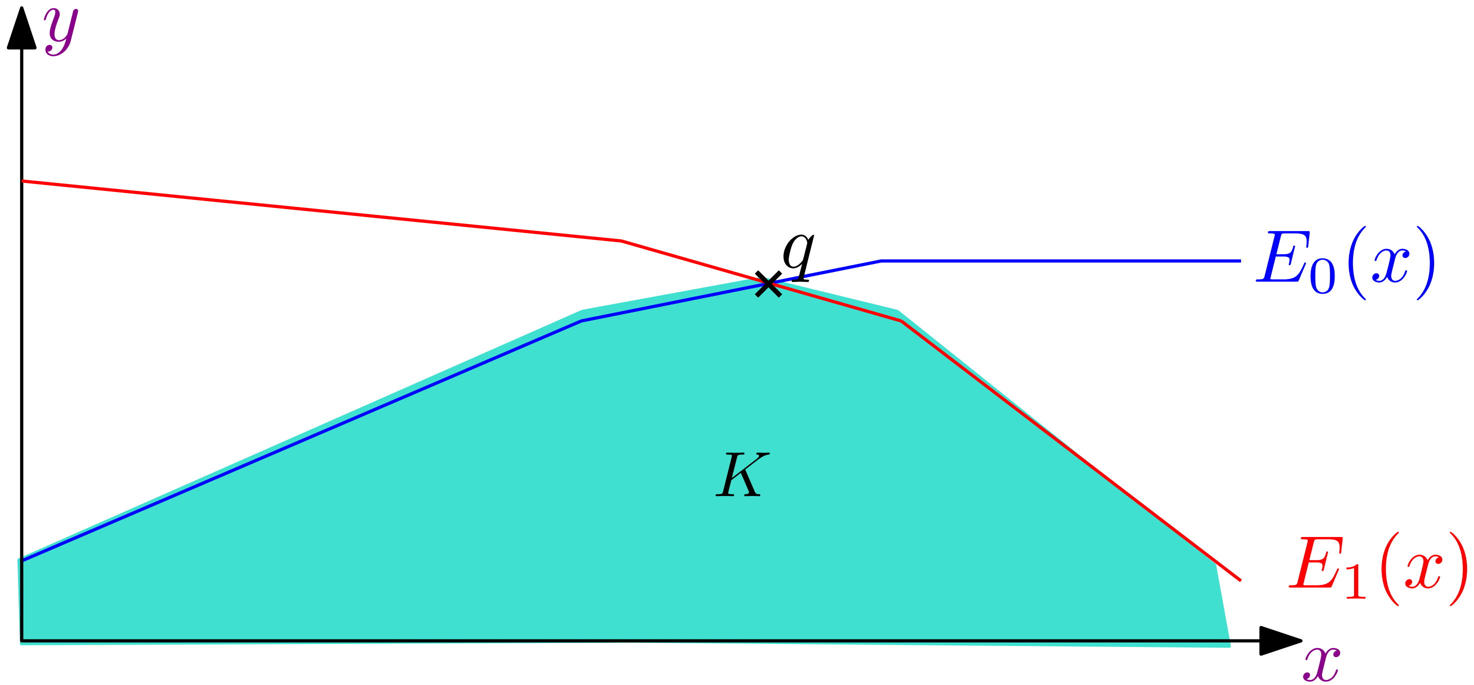
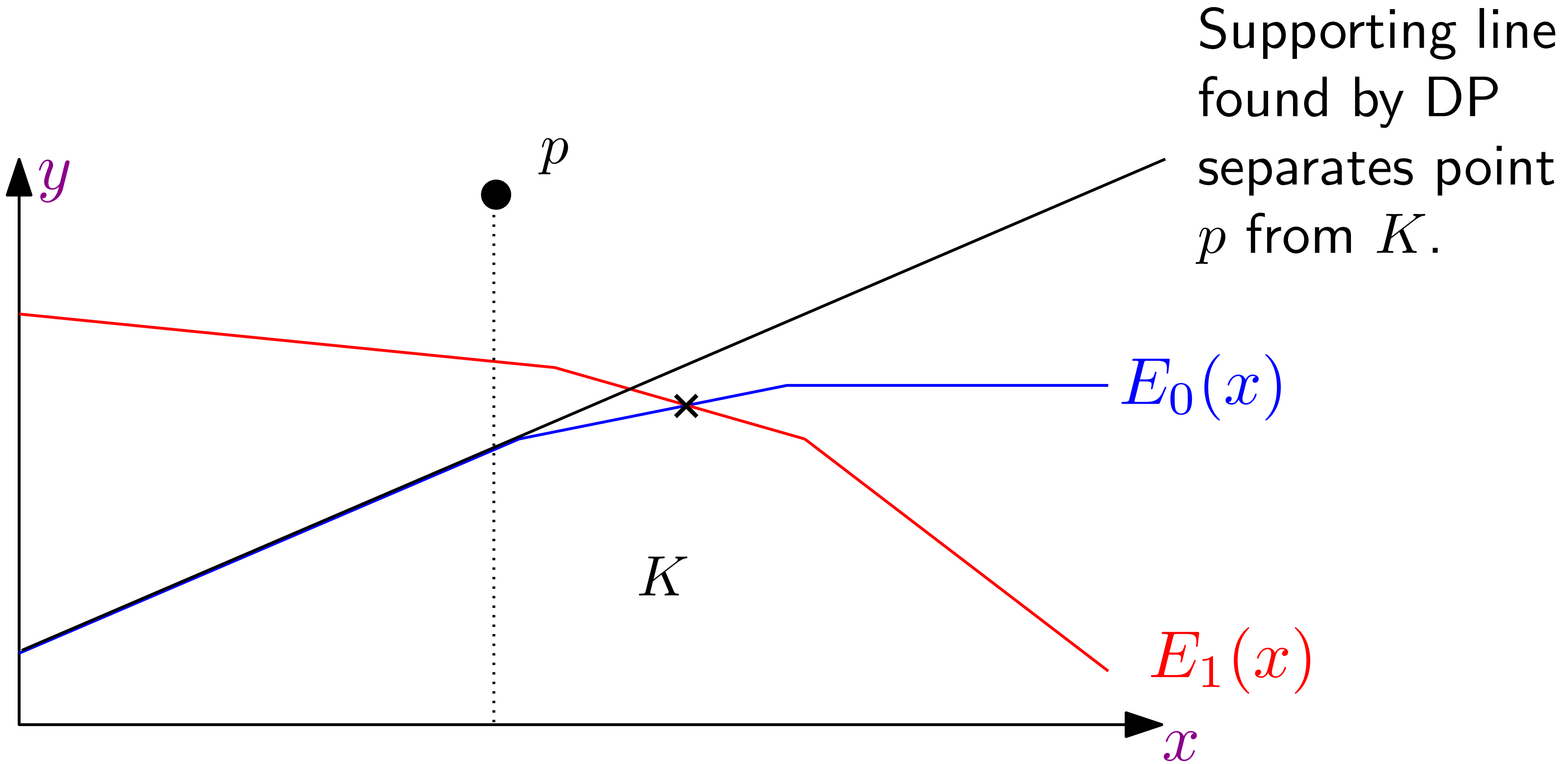


K is everything below *both* $E_0(x)$ and $E_1(x)$.
Want to find q, highest point in K.

- Where is the Separation Oracle?

- Where is the Separation Oracle?
- Known Dynamic Programming Algorithm!
  Returns the supporting lines of E_0 and E_1.
  Lower line either separates p from K, or proves that $p \in K$.



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  In m-ary case, AIFV-m codes construct m coding trees.
  Encoding/decoding switches between trees.
  Iterative algorithm for $m = 2$ case extends to general m case.
  Similar to $m = 2$, it was unknown how many iterations were needed.

  Binary searching technique can not be applied but ellipsoid technique can. Leads to $O(n^{2m+1} b)$ time algorithm.

- Details in the paper.

Outline

- Introduction
- AIFV-2 codes: cost and algorithm
- A Geometric Interpretation of the old algorithm
    - A New Binary Search Algorithm
    - An Ellipsoid Algorithm
- Extensions to AIFV-k codes (skip)
- Summing up and open questions

Summing up and open questions.

- Introduced idea of AIFV codes
- $O(n^5 b)$ for AIFV-2 codes is still high.
  Can this be improved?
  Best known so far is $O(n^4 b)$
- Are there *strongly polynomial* algorithms?
- Are there better AIFV codes?
  What is the tradeoff between number of coding trees used and compression? Everything known so far is empirical.