



The extended codes of projective two-weight codes [☆]

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ABSTRACT

A kind of extended codes of linear codes $\mathcal{C}$ with respect to a vector $\mathbf{u}$, denoted by $\overline{\mathcal{C}}(\mathbf{u})$, was introduced by Sun, Ding and Chen (2024) [58]. In this paper, we investigate the weight distribution of the extended codes $\overline{\mathcal{C}}(\mathbf{u})$ of projective two-weight codes $\mathcal{C}$ over finite fields. As a by-product, several families of linear codes with a few weights are presented, and six infinite families of distance-optimal linear codes are constructed in this paper.

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1. Introduction, motivations and objectives

Let q be a prime power, $\text{GF}(q)$ be the finite field with q elements. A q -ary $[n, k, d]$ linear code $\mathcal{C}$ is a k -dimensional linear subspace of $\text{GF}(q)^n$ with minimum distance d . A q -ary $[n, k, d]$ linear code is said to be distance-optimal if there is no q -ary $[n, k, d']$ linear code with $d' > d$. Let A_i denote the number of the codewords of Hamming weight i in $\mathcal{C}$. The sequence $(1, A_1, A_2, \dots, A_n)$ is called the *weight distribution* of $\mathcal{C}$ and the polynomial $\sum_{i=0}^n A_i z^i$ is referred to as the weight enumerator of $\mathcal{C}$. If $|\{i : A_i \neq 0, 1 \leq i \leq n\}| = t$, then the code $\mathcal{C}$ is called a *t-weight* linear code. Linear codes with a few weights are interesting because they are closely related to strongly regular graphs, secret sharing schemes, association schemes, and so on (see [5–7,11,16,18,51]).

For any q -ary $[n, k, d]$ linear code $\mathcal{C}$, there are two types of extended codes which are defined as follows.

- Let $\mathbf{u} \in \text{GF}(q)^n$ be a nonzero vector. The *extended code* of $\mathcal{C}$ with respect to $\mathbf{u}$ is defined by

$$\overline{\mathcal{C}}(\mathbf{u}) := \{(\mathbf{c}, c_{n+1}) : \mathbf{c} \in \mathcal{C}, c_{n+1} = \mathbf{c}\mathbf{u}^T\},$$

where $\mathbf{u}^T$ denotes the transpose of $\mathbf{u}$. When $\mathbf{u} = -\mathbf{1}$, where $\mathbf{1}$ denotes the all-one vector over $\text{GF}(q)$ with length n , the extended code $\overline{\mathcal{C}}(\mathbf{u})$ is called the *standardly extended code*.

- Let $\mathbf{G}$ be an $\ell \times n$ matrix whose row vectors linearly span $\mathcal{C}$ and $\mathbf{g} \in \text{GF}(q)^\ell$. The *extended code* of $\mathcal{C}$ with respect to $\mathbf{G}$ and $\mathbf{g}$, denoted by $\underline{\mathcal{C}}(\mathbf{G}, \mathbf{g})$, is a q -ary linear code linearly spanned by the row vectors of the matrix $(\mathbf{G} \ \mathbf{g}^T)$.

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For these two types of extended codes, we have the following.

- The code $\overline{\mathcal{C}}(\mathbf{u})$ has parameters $[n+1, k, \bar{d}]$, where $\bar{d} \in \{d, d+1\}$. Even if two codes $\overline{\mathcal{C}}(\mathbf{u}_1)$ and $\overline{\mathcal{C}}(\mathbf{u}_2)$ have the same minimum distance, they may have different weight distributions and different automorphism groups [58]. The standardly extended codes $\overline{\mathcal{C}}(\mathbf{u})$ of a few infinite families of linear codes were studied in [1–3,10,14,52,57]. Recently, nonstandardly extended codes of certain families of linear codes have been studied in [58]. In general, determining the parameters of such extended codes is difficult.
- The code $\underline{\mathcal{C}}(\mathbf{G}, \mathbf{g})$ has parameters $[n+1, \underline{k}, \underline{d}]$, where $\underline{k} \in \{k, k+1\}$ and $\underline{d} \in \{d, d+1\}$, where k and d denote the dimension and minimum distance of the linear code $\mathcal{C}$ linearly spanned by the row vectors of $\mathbf{G}$, respectively. In particular, if $\text{rank}(\mathbf{G})$ is equal to the number of rows of $\mathbf{G}$ (i.e., $\mathbf{G}$ is a generator matrix of the code $\mathcal{C}$), then there is a vector $\mathbf{u} \in \text{GF}(q)^n$ such that $\mathbf{G}\mathbf{u}^T = \mathbf{g}^T$ and $\underline{\mathcal{C}}(\mathbf{u}) = \underline{\mathcal{C}}(\mathbf{G}, \mathbf{g})$ (see [58]). Let $\mathcal{C}$ be a q -ary $[n, k, d]$ linear code. If there are $\mathbf{G}$ and $\mathbf{g}$ such that the row vectors of $\mathbf{G}$ linearly span $\mathcal{C}$ and $\underline{\mathcal{C}}(\mathbf{u})$ has parameters $[n+1, k, d+1]$, then $\mathcal{C}$ is said to be *extendable*. The extendability of certain linear codes $\mathcal{C}$ with respect to this extending technique $\underline{\mathcal{C}}(\mathbf{G}, \mathbf{g})$ and parameters of certain extended linear codes $\underline{\mathcal{C}}(\mathbf{u})$ were investigated in [8,24–26,31–36,38–49,53,54,56,64].

For differences and connections between the two extending techniques outlined above, the reader is referred to [58].

For a q -ary $[n, k, d]$ linear code $\mathcal{C}$, the dual code of $\mathcal{C}$ is defined by

$$\mathcal{C}^\perp := \left\{ (a_0, a_1, \dots, a_{n-1}) \in \text{GF}(q)^n : \sum_{i=0}^{n-1} a_i b_i = 0, \forall (b_0, b_1, \dots, b_{n-1}) \in \mathcal{C} \right\}.$$

It is well known that $\mathcal{C}^\perp$ is a q -ary $[n, n-k, d^\perp]$ linear code. If $d^\perp \geq 3$, $\mathcal{C}$ is said to be *projective*. Projective two-weight codes are an interesting family of linear codes because they are closely related to interesting combinatorial and geometric structures [6,11,18]. A number of projective two-weight codes have been constructed in [6,15,17,19,22,27–29,62,63]. But little is known about the extended codes $\overline{\mathcal{C}}(\mathbf{u})$ of projective two-weight codes $\mathcal{C}$.

This paper is motivated by the following open problem and [58].

Open Problem 1. What is the weight distribution of the extended code $\overline{\mathcal{C}}(\mathbf{u})$ for a given linear code $\mathcal{C}$ and vector $\mathbf{u}$?

As explained in [58], the determination of the minimum distance of an extended code $\overline{\mathcal{C}}(\mathbf{u})$ is in general a hard problem. For instance, the minimum distances of the extended codes $\overline{\mathcal{C}}(\mathbf{u})$ are not completely settled when $\mathcal{C}$ is a nonbinary Hamming code. Of course, Open Problem 1 is a much harder problem, compared with the determination of the minimum distance of $\overline{\mathcal{C}}(\mathbf{u})$. But Open Problem 1 could be settled for certain families of linear codes $\mathcal{C}$ and certain vectors $\mathbf{u}$.

The objectives of this paper are the following:

1. Determine the weight distributions of the extended codes $\overline{\mathcal{C}}(\mathbf{u})$ of projective two-weight codes $\mathcal{C}$.
2. Study the duals of the extended codes $\overline{\mathcal{C}}(\mathbf{u})$ of projective two-weight codes $\mathcal{C}$.

The rest of this paper is organized as follows. Section 2 presents some auxiliary results. Subsection 3.1 develops a general result about the weight enumerator of the extended codes of projective two-weight codes. Subsection 3.2 studies the extended codes of a family of constacyclic projective two-weight codes. Subsection 3.3 investigates the extended codes of ovoid codes. Subsection 3.4 considers the extended codes of the Denniston maximal arc codes. Section 4 summaries the contributions of this paper.

2. Some auxiliary results

Let $\mathcal{C}$ be a q -ary $[n, k, d]$ linear code with generator matrix $\mathbf{G}$, and $\mathbf{u} \in \text{GF}(q)^n$. Then $(\mathbf{G} \mathbf{G}\mathbf{u}^T)$ is a generator matrix of $\overline{\mathcal{C}}(\mathbf{u})$. Clearly, if $\mathbf{u} \in \mathcal{C}^\perp$, $d(\mathcal{C}(\mathbf{u})^\perp) = 1$. Such extended code $\overline{\mathcal{C}}(\mathbf{u})$ is said to be *trivial*. Below we always assume $\mathbf{u} \in \Pi(\mathcal{C}) := \text{GF}(q)^n \setminus \mathcal{C}^\perp$. Throughout this paper, $d(\mathcal{C})$ denotes the minimum distance of a linear code $\mathcal{C}$. For a q -ary code $\mathcal{C}$ with weight distribution $(1, A_1, \dots, A_n)$, we denote by $(1, A_1^\perp, \dots, A_n^\perp)$ the weight distribution of its dual code.

Lemma 2. [58] Let $\mathcal{C} \subseteq \text{GF}(q)^n$ be a linear code with $d(\mathcal{C}^\perp) \geq 3$ and let $\mathbf{u} \in \Pi(\mathcal{C})$. Then the following hold.

1. $d(\overline{\mathcal{C}}(\mathbf{u})^\perp) \geq 2$.
2. $d(\overline{\mathcal{C}}(\mathbf{u})^\perp) = 2$ if and only if there are $a \in \text{GF}(q)^*$ and $\mathbf{e}_j$ such that $\mathbf{u} + a\mathbf{e}_j \in \mathcal{C}^\perp$, where $\mathbf{e}_j$ is the vector of $\text{GF}(q)^n$ such that its j -th coordinator is 1 and its other coordinates are 0.
3. Let $\overline{A}_2$ denote the total number of codewords with weight 2 in $\overline{\mathcal{C}}(\mathbf{u})^\perp$. Then $\overline{A}_2^\perp \in \{0, q-1\}$.

The first three Pless power moment identities on the two weight distributions of a q -ary $[n, k]$ linear code are given as follows (see [30, Section 7.3]):

$$\begin{aligned}\sum_{i=0}^n A_i &= q^k, \\ \sum_{i=0}^n i A_i &= q^{k-1}[(q-1)n - A_1^\perp], \\ \sum_{i=0}^n i^2 A_i &= q^{k-2}\{(q-1)^2 n^2 + (q-1)n - [2(q-1)n - q + 2]A_1^\perp + 2A_2^\perp\}.\end{aligned}$$

These Pless power moment identities above will be needed later.

Let $\lambda \in \text{GF}(q) \setminus \{0\}$. A q -ary $[n, k, d]$ linear code $\mathcal{C}$ is λ -constacyclic if $(c_0, c_1, \dots, c_{n-1}) \in \mathcal{C}$ implies $(\lambda c_{n-1}, c_0, c_1, \dots, c_{n-2}) \in \mathcal{C}$. Define

$$\Phi: \text{GF}(q)^n \rightarrow \text{GF}(q)[x]/\langle x^n - \lambda \rangle, (c_0, c_1, \dots, c_{n-1}) \mapsto \sum_{i=0}^{n-1} c_i x^i.$$

It is known that every ideal of $\text{GF}(q)[x]/\langle x^n - \lambda \rangle$ is *principal* and a code $\mathcal{C} \subset \text{GF}(q)^n$ is λ -constacyclic if and only if $\Phi(\mathcal{C})$ is an ideal of $\text{GF}(q)[x]/\langle x^n - \lambda \rangle$. Due to this, we will identify $\mathcal{C}$ with $\Phi(\mathcal{C})$ for any q -ary λ -constacyclic code $\mathcal{C}$ of length n . Let $\mathcal{C} = \langle g(x) \rangle$ be a q -ary λ -constacyclic code of length n , where $g(x)$ is monic and has the smallest degree. Then $g(x)$ is called the *generator polynomial* and $h(x) = (x^n - \lambda)/g(x)$ is referred to as the *check polynomial* of $\mathcal{C}$.

Let $\lambda \in \text{GF}(q)^*$ with order r , and let $\text{ord}_{rn}(q)$ denote the multiplicative order of q modulo rn . Suppose $\gcd(n, q) = 1$ and $m = \text{ord}_{rn}(q)$. Then there is a primitive element α of $\text{GF}(q^m)$ such that $\beta = \alpha^{(q^m-1)/r}$ and $\beta^n = \lambda$. The minimal polynomial $\mathbb{M}_{\beta^i}(x)$ of β^i over $\text{GF}(q)$ is the monic polynomial of the smallest degree over $\text{GF}(q)$ with β^i as a zero. For any positive integer m , let $\text{Tr}_{q^m/q}$ denote the trace function from $\text{GF}(q^m)$ to $\text{GF}(q)$. The trace representation of λ -constacyclic codes is documented below (see [20], [55], [59, Theorem 1]).

Lemma 3. Let $\lambda \in \text{GF}(q)^*$ with $\text{ord}(\lambda) = r$. Let n be a positive integer such that $\gcd(n, q) = 1$. Let $m = \text{ord}_{rn}(q)$ and let $\beta \in \text{GF}(q^m)$ be a primitive rn -th root of unity such that $\beta^n = \lambda$. Let $\mathcal{C}$ be a q -ary λ -constacyclic code of length n with check polynomial $\prod_{j=1}^s \mathbb{M}_{\beta^{i_j}}(x)$. Then $\mathcal{C}$ has the trace representation

$$\mathcal{C} = \left\{ \left(\sum_{j=1}^s \text{Tr}_{q^m/q}(a_j \beta^{-ti_j}) \right)_{t=0}^{n-1} : a_j \in \text{GF}(q^{m_j}), 1 \leq j \leq s \right\},$$

where $m_j = \deg(\mathbb{M}_{\beta^{i_j}}(x))$.

3. The extended codes of projective two-weight codes

In this section, we study the extended codes of projective two-weight codes.

3.1. A general result about the weight enumerator of the extended codes of projective two-weight codes

The following theorem gives the weight enumerator of an extended code $\overline{\mathcal{C}}(\mathbf{u})$ of a projective two-weight code $\mathcal{C}$ in terms of the dimension and the nonzero weights in $\mathcal{C}$ and the number of codewords of weight 2 in $\overline{\mathcal{C}}(\mathbf{u})^\perp$.

Theorem 4. Let $\mathcal{C}$ be a q -ary projective two-weight code of length n and dimension $m \geq 3$ with nonzero weights w_1 and w_2 . Let $\mathbf{u} \in \Pi(\mathcal{C})$. Then the nonzero weights of $\overline{\mathcal{C}}(\mathbf{u})$ are from the set $\{w_1, w_1 + 1, w_2, w_2 + 1\}$. Furthermore,

$$\begin{aligned}\overline{A}_{w_1} &= \frac{(q^{m-1} - 1)w_2 - q^{m-2}(q-1)n + q^{m-2}\overline{A}_2^\perp}{w_2 - w_1}, \\ \overline{A}_{w_1+1} &= \frac{q^{m-1}(q-1)w_2 - q^{m-2}(q-1)^2n - q^{m-2}\overline{A}_2^\perp}{w_2 - w_1}, \\ \overline{A}_{w_2} &= \frac{(q^{m-1} - 1)w_1 - q^{m-2}(q-1)n + q^{m-2}\overline{A}_2^\perp}{w_1 - w_2}, \\ \overline{A}_{w_2+1} &= \frac{q^{m-1}(q-1)w_1 - q^{m-2}(q-1)^2n - q^{m-2}\overline{A}_2^\perp}{w_1 - w_2},\end{aligned}$$

where $\bar{A}_i$ denotes the total number of codewords with Hamming weight i in $\bar{\mathcal{C}}(\mathbf{u})$, and

$$\bar{A}_2^\perp = \begin{cases} q-1 & \text{if } \mathbf{u} + a\mathbf{e}_j \in \mathcal{C}^\perp \text{ for some } a \in \text{GF}(q)^* \text{ and } 1 \leq j \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since $\mathcal{C}$ has only nonzero weights w_1 and w_2 , the nonzero weights of $\bar{\mathcal{C}}(\mathbf{u})$ must belong to the set $\{w_1, w_1 + 1, w_2, w_2 + 1\}$. Let A_i denote the total number of codewords with Hamming weight i in $\mathcal{C}$. By assumption, $d(\mathcal{C}^\perp) \geq 3$. By the first three Pless power moment identities, we get

$$A_{w_1} + A_{w_2} = q^m - 1, \quad (1)$$

$$w_1 A_{w_1} + w_2 A_{w_2} = q^{m-1}(q-1)n, \quad (2)$$

$$w_1^2 A_{w_1} + w_2^2 A_{w_2} = q^{m-2}(q-1)n(qn - n + 1).$$

It is easily seen that

$$\bar{A}_{w_1} + \bar{A}_{w_1+1} = A_{w_1} \quad (3)$$

and

$$\bar{A}_{w_2} + \bar{A}_{w_2+1} = A_{w_2}. \quad (4)$$

Since $d(\mathcal{C}^\perp) \geq 3$, it follows from Lemma 2 that $d(\bar{\mathcal{C}}(\mathbf{u})^\perp) \geq 2$. Let $\bar{A}_2^\perp$ denote the total number of codewords with weight 2 in $\bar{\mathcal{C}}(\mathbf{u})^\perp$. By Lemma 2, $\bar{A}_2^\perp \in \{0, q-1\}$, and $\bar{A}_2^\perp = q-1$ if and only if $\mathbf{u} + a\mathbf{e}_j \in \mathcal{C}^\perp$ for some $a \in \text{GF}(q)^*$ and $1 \leq j \leq n$.

By the second Pless power moment identity, we have

$$\begin{aligned} & w_1 \bar{A}_{w_1} + (w_1 + 1) \bar{A}_{w_1+1} + w_2 \bar{A}_{w_2} + (w_2 + 1) \bar{A}_{w_2+1} \\ &= w_1 (\bar{A}_{w_1} + \bar{A}_{w_1+1}) + w_2 (\bar{A}_{w_2} + \bar{A}_{w_2+1}) + \bar{A}_{w_1+1} + \bar{A}_{w_2+1} \\ &= w_1 A_{w_1} + w_2 A_{w_2} + \bar{A}_{w_1+1} + \bar{A}_{w_2+1} \\ &= q^{m-1}(q-1)n + \bar{A}_{w_1+1} + \bar{A}_{w_2+1} \\ &= q^{m-1}(q-1)(n+1). \end{aligned} \quad (5)$$

It follows from (5) that

$$\bar{A}_{w_1+1} + \bar{A}_{w_2+1} = q^{m-1}(q-1). \quad (6)$$

By the third Pless power moment identity, we get

$$\begin{aligned} & w_1^2 \bar{A}_{w_1} + (w_1 + 1)^2 \bar{A}_{w_1+1} + w_2^2 \bar{A}_{w_2} + (w_2 + 1)^2 \bar{A}_{w_2+1} \\ &= w_1^2 (\bar{A}_{w_1} + \bar{A}_{w_1+1}) + w_2^2 (\bar{A}_{w_2} + \bar{A}_{w_2+1}) + 2(w_1 \bar{A}_{w_1+1} + w_2 \bar{A}_{w_2+1}) + \bar{A}_{w_1+1} + \bar{A}_{w_2+1} \\ &= w_1^2 A_{w_1} + w_2^2 A_{w_2} + 2(w_1 \bar{A}_{w_1+1} + w_2 \bar{A}_{w_2+1}) + \bar{A}_{w_1+1} + \bar{A}_{w_2+1} \\ &= q^{m-2}(q-1)n(qn - n + 1) + q^{m-1}(q-1) + 2(w_1 \bar{A}_{w_1+1} + w_2 \bar{A}_{w_2+1}) \\ &= q^{m-2}[(q-1)(n+1)(qn + 1) - (n+1) + 1] + 2\bar{A}_2^\perp. \end{aligned} \quad (7)$$

It follows from (7) that

$$w_1 \bar{A}_{w_1+1} + w_2 \bar{A}_{w_2+1} = q^{m-2}(q-1)^2 n + q^{m-2} \bar{A}_2^\perp. \quad (8)$$

Combining (6) and (8), we get

$$\bar{A}_{w_1+1} = \frac{q^{m-1}(q-1)w_2 - q^{m-2}(q-1)^2 n - q^{m-2} \bar{A}_2^\perp}{w_2 - w_1}, \quad (9)$$

$$\bar{A}_{w_2+1} = \frac{q^{m-1}(q-1)w_1 - q^{m-2}(q-1)^2 n - q^{m-2} \bar{A}_2^\perp}{w_1 - w_2}. \quad (10)$$

Combining (1) and (2), we have

$$A_{w_1} = \frac{(q^m - 1)w_2 - q^{m-1}(q-1)n}{w_2 - w_1}, \quad (11)$$

$$A_{w_2} = \frac{(q^m - 1)w_1 - q^{m-1}(q-1)n}{w_1 - w_2}. \quad (12)$$

Combining (3), (9) and (11), we obtain

$$\overline{A}_{w_1} = \frac{(q^{m-1} - 1)w_2 - q^{m-2}(q-1)n + q^{m-2}\overline{A}_2^\perp}{w_2 - w_1}.$$

Combining (4), (10) and (12), we arrive at

$$\overline{A}_{w_2} = \frac{(q^{m-1} - 1)w_1 - q^{m-2}(q-1)n + q^{m-2}\overline{A}_2^\perp}{w_1 - w_2}.$$

This completes the proof. $\square$

Corollary 5. Let $\mathcal{C}$ be a q -ary projective two-weight code of length n and dimension $m \geq 6$. Suppose $n \neq (q^m - q)/(q - 1)$ and $\mathbf{u} \in \Pi(\mathcal{C})$. Then $d(\overline{\mathcal{C}}(\mathbf{u})) = d(\mathcal{C})$.

Proof. Let $0 < w_1 < w_2$ be the two weights of $\mathcal{C}$. By Theorem 4, to prove the desired result, we only need to prove

$$(q^{m-1} - 1)w_2 - q^{m-2}(q-1)(n - \varepsilon) \neq 0$$

for $\varepsilon \in \{0, 1\}$. Suppose on the contrary that

$$(q^{m-1} - 1)w_2 - q^{m-2}(q-1)(n - \varepsilon) = 0.$$

It then follows that $q^{m-2} \mid w_2$. Let $w_2 = \lambda q^{m-2}$ for some λ . Then $n = \lambda(\frac{q^{m-1}-1}{q-1}) + \varepsilon$. On the other hand,

$$A_{w_1} + A_{w_2} = q^m - 1, \quad (13)$$

$$w_1 A_{w_1} + w_2 A_{w_2} = q^{m-1}(q-1)n, \quad (14)$$

$$w_1^2 A_{w_1} + w_2^2 A_{w_2} = q^{m-2}(q-1)n(qn - n + 1). \quad (15)$$

By (13) $\times w_2 - (14)$, we get

$$(w_2 - w_1)A_{w_1} = q^{m-2}(q-1)(\lambda - q\varepsilon). \quad (16)$$

By (14) $\times w_2 - (15)$, we obtain

$$w_1(w_2 - w_1)A_{w_1} = q^{m-2}(q-1)n[\lambda - 1 - (q-1)\varepsilon]. \quad (17)$$

Combining (16) and (17), we have

$$w_1 = \frac{n[\lambda - 1 - (q-1)\varepsilon]}{\lambda - q\varepsilon}.$$

If $\varepsilon = 1$, we have $w_1 = n$, a contradiction. If $\varepsilon = 0$, then $w_1 = (\lambda - 1)(\frac{q^{m-1}-1}{q-1})$. Notice that $0 < w_1 < w_2$ and $n \neq (q^m - q)/(q - 1)$, we get $2 \leq \lambda \leq q - 1$. By (16), we get

$$\lambda(q-1)q^{m-2} \equiv 0 \pmod{w_2 - w_1}. \quad (18)$$

Notice that $w_2 - w_1 = \frac{q^{m-1}-1}{q-1} - \lambda(\frac{q^{m-2}-1}{q-1})$, we have $\gcd(w_2 - w_1, q) = \gcd(\lambda - 1, q) < q$. By (18), we obtain

$$\frac{w_2 - w_1}{\gcd(\lambda - 1, q)} \mid \lambda(q-1).$$

It then follows that $w_2 - w_1 < \lambda(q-1)\gcd(\lambda - 1, q)$. Since $m \geq 6$ and $\lambda \leq q - 1$,

$$w_2 - w_1 > q^3 > (q-1)^2(q-2) \geq \lambda(q-1)\gcd(\lambda - 1, q),$$

a contradiction. This completes the proof. $\square$

In the next three subsections, we will make use of Theorem 4 to study the extended codes of several infinite families of projective two-weight codes.

3.2. The extended codes of a family of constacyclic projective two-weight codes

Let $q > 2$ be a prime power and $m = 2ls \geq 4$, where l and s are positive integers. Let h be a positive divisor of $q^s + 1$ and $2 \leq h < q^{ls} + 1$. Let α be a primitive element of $\text{GF}(q^m)$ and $\beta = \alpha^h$. Then β is a primitive $(q^m - 1)/h$ -th root of unity. Let $n = \frac{q^m - 1}{h(q - 1)}$ and $\lambda = \beta^n$. Then λ is a primitive element of $\text{GF}(q)$. Since $q > 2$, $\lambda \neq 1$. Let $\mathbb{M}_\beta(x)$ denote the minimal polynomial of β over $\text{GF}(q)$, and let $\mathcal{C}(\beta)$ be the q -ary λ -constacyclic code of length n with check polynomial $\mathbb{M}_\beta(x)$. Let $\theta = \beta^{-1}$. Then $\mathcal{C}(\beta)$ has the following trace representation:

$$\{\mathbf{c}(a) = (\text{Tr}_{q^m/q}(a), \text{Tr}_{q^m/q}(a\theta), \dots, \text{Tr}_{q^m/q}(a\theta^{n-1})) : a \in \text{GF}(q^m)\}.$$

This family of constacyclic codes was investigated in [17]. In this subsection, we study the extended codes of this family of constacyclic codes.

Theorem 6. [17] *The λ -constacyclic code $\mathcal{C}(\beta)$ has parameters $[n, m]$ and weight enumerator*

$$1 + \frac{q^m - 1}{h} z^{\frac{q^{m+(-1)^l(h-1)q^{m/2}}}{qh}} + \frac{(h-1)(q^m - 1)}{h} z^{\frac{q^{m+(-1)^{l-1}q^{m/2}}}{qh}}.$$

The dual code $\mathcal{C}(\beta)^\perp$ has parameters $[n, n - m, d \geq 3]$.

Theorem 7. *Let $\mathcal{C}(\beta)$ be the λ -constacyclic code in Theorem 6 and $\mathbf{u} \in \Pi(\mathcal{C}(\beta))$. Then $d(\overline{\mathcal{C}(\beta)}(\mathbf{u})) = d(\mathcal{C}(\beta))$.*

Proof. Let $w_1 = \frac{q^{m+(-1)^l(h-1)q^{m/2}}}{qh}$ and $w_2 = \frac{q^{m+(-1)^{l-1}q^{m/2}}}{qh}$. According to Theorem 4, we deduce

$$\overline{A}_{w_1} = \left[q^{(m-2)/2} + (-1)^l \right] \frac{q^{m/2} - (-1)^l}{h} - q^{(m-2)/2} (-1)^l \overline{A}_2^\perp, \quad (19)$$

$$\overline{A}_{w_1+1} = q^{(m-2)/2} \left[\frac{q^{m/2} - (-1)^l}{h} (q - 1) + (-1)^l \overline{A}_2^\perp \right], \quad (20)$$

$$\overline{A}_{w_2} = q^{m-1} - 1 - [q^{(m-2)/2} + (-1)^l] \frac{q^{m/2} - (-1)^l}{h} + q^{(m-2)/2} (-1)^l \overline{A}_2^\perp, \quad (21)$$

$$\overline{A}_{w_2+1} = q^{(m-2)/2} \left[\frac{(h-1)q^{m/2} + (-1)^l}{h} (q - 1) + (-1)^{l-1} \overline{A}_2^\perp \right], \quad (22)$$

where $\overline{A}_2^\perp$ denotes the total number of codewords with weight 2 in $\overline{\mathcal{C}(\beta)}(\mathbf{u})^\perp$ and $\overline{A}_2^\perp \in \{0, q - 1\}$.

Since $m \geq 4$, it follows from (19) that

$$\overline{A}_{w_1} \equiv \frac{(-1)^l q^{m/2} - 1}{h} \not\equiv 0 \pmod{q^{(m-2)/2}}.$$

Then $\overline{A}_{w_1} > 0$. Note that $h < q^{m/2} + 1$ and $\overline{A}_2^\perp \in \{0, q - 1\}$. It then follows from (20) and (22) that $\overline{A}_{w_1+1} > 0$ and $\overline{A}_{w_2+1} > 0$.

When l is even, it follows from (21) that

$$\begin{aligned} \overline{A}_{w_2} &= q^{m-1} - 1 - [q^{(m-2)/2} + 1] \frac{q^{m/2} - 1}{h} + q^{(m-2)/2} \overline{A}_2^\perp \\ &\geq q^{m-1} - 1 - [q^{(m-2)/2} + 1] \frac{q^{m/2} - 1}{2} \\ &= \frac{(q^{m/2} + 1)(q^{(m-2)/2} - 1)}{2} > 0. \end{aligned}$$

When l is odd, it follows from (21) that

$$\begin{aligned} \overline{A}_{w_2} &= q^{m-1} - 1 - [q^{(m-2)/2} - 1] \frac{q^{m/2} + 1}{h} - q^{(m-2)/2} \overline{A}_2^\perp \\ &\geq q^{m-1} - 1 - [q^{(m-2)/2} - 1] \frac{q^{m/2} + 1}{2} - q^{(m-2)/2} (q - 1) \\ &= \frac{(q^{m/2} + 1)(q^{(m-2)/2} - 1)}{2} > 0. \end{aligned}$$

Consequently, $d(\overline{\mathcal{C}(\beta)}(\mathbf{u})) = d(\mathcal{C}(\beta))$. This completes the proof. $\square$

Theorem 7 has proved that $d(\overline{\mathcal{C}(\beta)}(\mathbf{u})) = d(\mathcal{C}(\beta))$. The weight distribution of $\overline{\mathcal{C}(\beta)}(\mathbf{u})$ depends on β and $\mathbf{u}$ in general. Now we consider the weight distribution of the standardly extended code $\overline{\mathcal{C}(\beta)}(-\mathbf{1})$.

Lemma 8. Let $\mathcal{C}(\beta)$ be the λ -constacyclic code in Theorem 6. Then $d(\overline{\mathcal{C}(\beta)}(-\mathbf{1})^\perp) \geq 2$. Let $\overline{A}_2^\perp$ denote the total number of codewords with weight 2 in $\overline{\mathcal{C}(\beta)}(-\mathbf{1})^\perp$. Then

$$\overline{A}_2^\perp = \begin{cases} 0 & \text{if } \theta - 1 \notin \langle \theta \rangle, \\ q - 1 & \text{if } \theta - 1 \in \langle \theta \rangle, \end{cases}$$

where $\theta = \beta^{-1}$.

Proof. Let $\mathbb{M}_\theta(x)$ be the minimal polynomial of θ over $\text{GF}(q)$. Then $\mathcal{C}(\beta)^\perp = \langle \mathbb{M}_\theta(x) \rangle$ is a q -ary λ^{-1} -constacyclic code of length n . Suppose $-\mathbf{1} \in \mathcal{C}(\beta)^\perp$. Then

$$1 + x + \cdots + x^{n-1} \equiv 0 \pmod{\mathbb{M}_\theta(x)}.$$

It follows that

$$1 + \theta + \cdots + \theta^{n-1} = \frac{\theta^n - 1}{\theta - 1} = \frac{\lambda^{-1} - 1}{\theta - 1} = 0,$$

a contradiction. By Theorem 2, $d(\overline{\mathcal{C}(\beta)}(-\mathbf{1})^\perp) \geq 2$.

Suppose $d(\overline{\mathcal{C}(\beta)}(-\mathbf{1})^\perp) = 2$. By Theorem 2, there are $a \in \text{GF}(q)^*$ and $\mathbf{e}_j$ such that $-\mathbf{1} + a\mathbf{e}_j \in \mathcal{C}(\beta)^\perp$. It follows that

$$1 + x + \cdots + x^{n-1} - ax^{j-1} \equiv 0 \pmod{\mathbb{M}_\theta(x)}. \quad (23)$$

It is clear that (23) is equivalent to

$$1 + \theta + \cdots + \theta^{n-1} - a\theta^{j-1} = \frac{\lambda^{-1} - 1}{\theta - 1} - a\theta^{j-1} = 0 \iff \theta^j - \theta^{j-1} = b,$$

where $b = a^{-1}(\lambda^{-1} - 1) \in \text{GF}(q)^*$. Note that θ^n is a primitive element of $\text{GF}(q)$. If there is an integer j such that $\theta^j - \theta^{j-1} \in \text{GF}(q)^*$, then $\theta - 1 \in \langle \theta \rangle$. Therefore, if $\theta - 1 \notin \langle \theta \rangle$, we have $\overline{A}_2^\perp = 0$.

Suppose $\theta - 1 \in \langle \theta \rangle$. Then there is a unique integer i with $0 \leq i \leq (q-1)n-1$ such that $\theta - 1 = \theta^i$. Suppose $i = ni_0 + i_1$, where $0 \leq i_1 \leq n-1$, $0 \leq i_0 \leq q-2$. Then $\theta - 1 = \theta^{ni_0\theta^{i_1}}$. We claim $2 \leq i_1 \leq n-1$. If $i_1 = 0$, then $\theta - 1 \in \text{GF}(q)$, a contradiction. If $i_1 = 1$, then $(1 - \theta^{ni_0})\theta = 1$. Note that $1 - \theta^{ni_0} \in \text{GF}(q)^*$, then $\theta \in \text{GF}(q)^*$, a contradiction. For any $b \in \text{GF}(q)^*$, there is a unique integer ℓ with $0 \leq \ell \leq q-2$ such that $b = \theta^{n\ell}$. Consequently,

$$\begin{aligned} \theta^j - \theta^{j-1} = b &\iff \theta^{j-1} = \theta^{n(\ell-i_0-1)+n-i_1} \\ &\iff j-1 \equiv n(\ell-i_0-1) + n-i_1 \pmod{(q-1)n} \\ &\iff j-1 = n-i_1, \text{ and } \ell \equiv i_0+1 \pmod{q-1}. \end{aligned}$$

Therefore, there is a unique pair (b, j) such that $\theta^j - \theta^{j-1} = b$. The desired value of $\overline{A}_2^\perp$ follows from Lemma 2. This completes the proof. $\square$

Corollary 9. Let $\mathcal{C}(\beta)$ be the λ -constacyclic code in Theorem 6. Then $d(\overline{\mathcal{C}(\beta)}(-\mathbf{1})) = d(\mathcal{C}(\beta))$ and $\overline{\mathcal{C}(\beta)}(-\mathbf{1})$ is a 4-weight code.

Proof. By Lemma 8, $-\mathbf{1} \notin \mathcal{C}^\perp$. The desired results follow from Theorem 7. $\square$

Corollary 9 proves that $d(\overline{\mathcal{C}(\beta)}(-\mathbf{1})) = d(\mathcal{C}(\beta))$ and the weight distribution of the standardly extended code $\overline{\mathcal{C}(\beta)}(-\mathbf{1})$ is related to β in general. Let $\beta' = \beta^v$, where v is a positive integer with $2 \leq v \leq \frac{q^m-1}{h} - 1$ and $\gcd(v, \frac{q^m-1}{h}) = 1$. Then β' is also a primitive $\frac{q^m-1}{h}$ -th root of unity. It is easily verified that $\mathcal{C}(\beta')$ is equivalent to $\mathcal{C}(\beta)$. However, the weight distribution of $\overline{\mathcal{C}(\beta)}(-\mathbf{1})$ and $\overline{\mathcal{C}(\beta')}(-\mathbf{1})$ may be different, so $\overline{\mathcal{C}(\beta)}(-\mathbf{1})$ and $\overline{\mathcal{C}(\beta')}(-\mathbf{1})$ may not be equivalent. Two examples are given to illustrate this.

Example 10. Let $(q, l, s, h) = (3, 2, 1, 2)$. Then $m = 2ls = 4$ and $n = \frac{q^m-1}{h(q-1)} = 20$. Let α be a primitive element of $\text{GF}(q^m)$ with $\alpha^4 + 2\alpha^3 + 2 = 0$. The following hold.

1. Let $\beta = \alpha^h$. Then $\mathbb{M}_\beta(x) = x^4 + 2x^3 + x^2 + 1$. Let $\theta = \beta^{-1}$. It is easily verified that $\theta - 1 \notin \langle \theta \rangle$. Let $\mathcal{C}(\beta)$ be the negacyclic code of length n with check polynomial $\mathbb{M}_\beta(x)$. Then $\mathcal{C}(\beta)$ has parameters $[20, 4, 12]$ and weight enumerator $1 + 40z^{12} + 40z^{15}$. The dual code $\mathcal{C}(\beta)^\perp$ has parameters $[20, 16, 3]$. The standardly extended code $\overline{\mathcal{C}(\beta)}(-1)$ has parameters $[21, 4, 12]$ and weight enumerator $1 + 10z^{12} + 30z^{13} + 16z^{15} + 24z^{16}$. The dual code $\overline{\mathcal{C}(\beta)}(-1)^\perp$ has parameters $[21, 17, 3]$. Moreover, $\overline{\mathcal{C}(\beta)}(-1)$ and $\overline{\mathcal{C}(\beta)}(-1)^\perp$ are distance-optimal codes [21].
2. Let $\beta = \alpha^{7h}$. Then $\mathbb{M}_\beta(x) = x^4 + x^3 + x^2 + 1$. Let $\theta = \beta^{-1}$. It is easily verified that $\theta - 1 \in \langle \theta \rangle$. Let $\mathcal{C}(\beta)$ be the negacyclic code of length n with check polynomial $\mathbb{M}_\beta(x)$. Then $\mathcal{C}(\beta)$ has parameters $[20, 4, 12]$ and weight enumerator $1 + 40z^{12} + 40z^{15}$. The dual code $\mathcal{C}(\beta)^\perp$ has parameters $[20, 16, 3]$. The standardly extended code $\overline{\mathcal{C}(\beta)}(-1)$ has parameters $[21, 4, 12]$ and weight enumerator $1 + 16z^{12} + 24z^{13} + 10z^{15} + 30z^{16}$. The dual code $\overline{\mathcal{C}(\beta)}(-1)^\perp$ has parameters $[21, 17, 2]$.

Example 11. Let $(q, l, s, h) = (5, 2, 1, 3)$. Then $m = 2ls = 4$ and $n = \frac{q^m - 1}{h(q-1)} = 52$. Let α be a primitive element of $\text{GF}(q^m)$ with $\alpha^4 + 4\alpha^2 + 4\alpha + 2 = 0$. The following hold.

1. Let $\beta = \alpha^h$. Then $\mathbb{M}_\beta(x) = x^4 + 2x^3 + 3x^2 + 3x + 3$. Let $\theta = \beta^{-1}$. It is easily verified that $\theta - 1 \notin \langle \theta \rangle$. Let $\mathcal{C}(\beta)$ be the 2-constacyclic code of length n with check polynomial $\mathbb{M}_\beta(x)$. Then $\mathcal{C}(\beta)$ has parameters $[52, 4, 40]$ and weight enumerator $1 + 416z^{40} + 208z^{45}$. The dual code $\mathcal{C}(\beta)^\perp$ has parameters $[52, 48, 3]$. The standardly extended code $\overline{\mathcal{C}(\beta)}(-1)$ has parameters $[53, 4, 40]$ and weight enumerator $1 + 76z^{40} + 340z^{41} + 48z^{45} + 160z^{46}$. The dual code $\overline{\mathcal{C}(\beta)}(-1)^\perp$ has parameters $[53, 49, 3]$. Moreover, $\overline{\mathcal{C}(\beta)}(-1)$ and $\overline{\mathcal{C}(\beta)}(-1)^\perp$ are distance-optimal codes [21].
2. Let $\beta = \alpha^{21h}$. Then $\mathbb{M}_\beta(x) = x^4 + 3x^3 + 3x^2 + 2x + 3$. Let $\theta = \beta^{-1}$. It is easily verified that $\theta - 1 \in \langle \theta \rangle$. Let $\mathcal{C}(\beta)$ be the 2-constacyclic code of length n with check polynomial $\mathbb{M}_\beta(x)$. Then $\mathcal{C}(\beta)$ has parameters $[52, 4, 40]$ and weight enumerator $1 + 416z^{40} + 208z^{45}$. The dual code $\mathcal{C}(\beta)^\perp$ has parameters $[52, 48, 3]$. The standardly extended code $\overline{\mathcal{C}(\beta)}(-1)$ has parameters $[53, 4, 40]$ and weight enumerator $1 + 96z^{40} + 320z^{41} + 28z^{45} + 180z^{46}$. The dual code $\overline{\mathcal{C}(\beta)}(-1)^\perp$ has parameters $[53, 29, 2]$.

It looks hard to completely determine the weight distribution of the standardly extended code $\overline{\mathcal{C}(\beta)}(-1)$. When $m = 4$ and $h = q + 1$, it is easily seen that $\mathcal{C}(\beta)$ has parameters $[q^2 + 1, 4, q^2 - q]$. The weight distribution of $\overline{\mathcal{C}(\beta)}(-1)$ will be determined in Theorem 20.

3.3. The extended codes of ovoid codes

Throughout this subsection, let $q > 2$ be a prime power and let $n = q^2 + 1$.

3.3.1. Some extended codes of the elliptic quadric codes and Tits ovoid codes

An ovoid in the projective space $\text{PG}(3, \text{GF}(q))$ is a set of $q^2 + 1$ points such that no three of them are colinear (i.e., on the same line). A classical ovoid $\mathcal{V}$ can be defined as the set of all points given by

$$\mathcal{V} = \{(0, 0, 1, 0)\} \cup \{(x, y, x^2 + xy + ay^2, 1) : x, y \in \text{GF}(q)\},$$

where $a \in \text{GF}(q)$ is such that the polynomial $x^2 + x + a$ has no root in $\text{GF}(q)$. Such ovoid is called an *elliptic quadric*, as the points come from a non-degenerate elliptic quadratic form. When q is odd, every ovoid in $\text{PG}(3, \text{GF}(q))$ is equivalent to the elliptic quadric.

For $q = 2^{2e+1}$ with $e \geq 1$, there is an ovoid which is not an elliptic quadric and is called the *Tits ovoid*. It is defined by

$$\mathcal{T} = \{(0, 0, 1, 0)\} \cup \{(x, y, x^\sigma + xy + y^{\sigma+2}, 1) : x, y \in \text{GF}(q)\},$$

where $\sigma = 2^{e+1}$. It is known that the Tits ovoid is not equivalent to the elliptic quadric.

Let $\mathcal{S} = \{\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_{q^2+1}\}$ be a subset of $\text{PG}(3, \text{GF}(q))$ with $q^2 + 1$ elements, where $q > 2$. Let $\mathcal{C}_\mathcal{S}$ be the q -ary linear code with generator matrix

$$\mathbf{G}_\mathcal{S} = \begin{pmatrix} \mathbf{s}_1^\top & \mathbf{s}_2^\top & \cdots & \mathbf{s}_{q^2+1}^\top \end{pmatrix},$$

where $\mathbf{s}^\top$ denotes the transpose of $\mathbf{s}$. It is known that $\mathcal{S}$ is an ovoid if and only if $\mathcal{C}_\mathcal{S}$ has parameters $[q^2 + 1, 4, q^2 - q]$ (see [13]). Therefore, any q -ary $[q^2 + 1, 4, q^2 - q]$ code is called an *ovoid code*. Every ovoid can be used to construct an ovoid code and vice versa. The next theorem summarizes information on ovoid codes [18, Chapter 13].

Theorem 12. Let $q > 2$ be a prime power. Then the following statements are true.

- Every q -ary ovoid code must have parameters $[q^2 + 1, 4, q^2 - q]$ and weight enumerator

$$1 + (q^2 - q)(q^2 + 1)z^{q^2 - q} + (q - 1)(q^2 + 1)z^{q^2}.$$

When q is odd, all q -ary ovoid codes are monomially-equivalent. When q is even, the q -ary elliptic quadric code and the q -ary Tits ovoid code are not monomially-equivalent.

- The dual of every q -ary ovoid code must have parameters $[q^2 + 1, q^2 - 3, 4]$.
- The minimum weight codewords in a q -ary ovoid code support a 3 -($q^2 + 1, q^2 - q, (q - 2)(q^2 - q - 1)$) design and the complementary design of this design is a Steiner system $S(3, 1 + q, 1 + q^2)$. When q is odd, all these 3 -($q^2 + 1, q^2 - q, (q - 2)(q^2 - q - 1)$) designs are isomorphic. The minimum weight codewords in the dual of a q -ary ovoid code support a 3 -($q^2 + 1, 4, q - 2$) design.

The parameters of the extended codes of any q -ary ovoid codes are documented in the following theorem.

Theorem 13. Let $q \geq 4$ be a prime power. Let $\mathcal{C}$ be a q -ary ovoid code. Then $\overline{\mathcal{C}}(\mathbf{u})$ has parameters $[q^2 + 2, 4, q^2 - q]$ and is distance-optimal with respect to the Griesmer bound, for any $\mathbf{u} \in \text{GF}(q)^{q^2+1}$.

Proof. By definition, $\mathcal{C}$ has parameters $[q^2 + 1, 4, q^2 - q]$. It is easily verified that $\mathcal{C}$ meets the Griesmer bound. Note that

$$\sum_{i=0}^3 \left\lceil \frac{q^2 - q + 1}{q^i} \right\rceil = q^2 + 3.$$

There does not exist a q -ary $[q^2 + 2, 4, d]$ linear code with $d > q^2 - q$. The desired conclusions follow. $\square$

The parameters of the extended codes of the dual of a q -ary ovoid code are documented in the next theorem.

Theorem 14. Let $q \geq 4$ be a prime power. Let $\mathcal{C}$ be a q -ary ovoid code. Then $\overline{\mathcal{C}^\perp}(\mathbf{u})$ has parameters $[q^2 + 2, q^2 - 3, 4]$, and is distance-optimal, for any $\mathbf{u} \in \text{GF}(q)^{q^2+1}$.

Proof. It is clear that $\dim(\overline{\mathcal{C}^\perp}(\mathbf{u})) = \dim(\mathcal{C}^\perp) = q^2 - 3$. Since $d(\mathcal{C}^\perp) = 4$, it follows that $d(\overline{\mathcal{C}^\perp}(\mathbf{u})) \geq 4$. Suppose there exists a q -ary $[q^2 + 2, q^2 - 3, d \geq 5]$ linear code. By the sphere packing bound [30],

$$1 + \binom{q^2 + 2}{1}(q - 1) + \binom{q^2 + 2}{2}(q - 1)^2 \leq q^5.$$

It is easily verified that

$$1 + \binom{q^2 + 2}{1}(q - 1) + \binom{q^2 + 2}{2}(q - 1)^2 - q^5 = \frac{q^6 - 4q^5 + 4q^4 - 4q^3 + 3q^2}{2} > 0,$$

a contradiction. Therefore, a q -ary $[q^2 + 2, q^2 - 3, d \geq 5]$ linear code does not exist. The desired result follows. $\square$

Theorem 14 shows that the extended codes of any q -ary $[q^2 + 1, q^2 - 3, 4]$ linear codes have parameters $[q^2 + 2, q^2 - 3, 4]$. The parameters of the standardly extended codes of $\mathcal{C}_{\mathcal{V}}$ and $\mathcal{C}_{\mathcal{T}}$ are documented in the next theorem.

Theorem 15. Let $q > 2$ be a prime power and $S \in \{\mathcal{V}, \mathcal{T}\}$. Then the standardly extended code $\overline{\mathcal{C}_S}(-\mathbf{1})$ has parameters $[q^2 + 2, 4, q^2 - q]$, and weight enumerator

$$1 + q(q^2 - 1)z^{q^2 - q} + q^2(q - 1)^2z^{q^2 - q + 1} + (q - 1)z^{q^2} + q^2(q - 1)z^{q^2 + 1}.$$

The dual code $\overline{\mathcal{C}_S}(-\mathbf{1})^\perp$ has parameters $[q^2 + 2, q^2 - 2, 2]$.

Proof. It is easily verified that

$$\mathbf{G}_{\mathcal{V}}\mathbf{1}^T = \begin{pmatrix} \sum_{x \in \text{GF}(q)} \sum_{y \in \text{GF}(q)} x \\ \sum_{x \in \text{GF}(q)} \sum_{y \in \text{GF}(q)} y \\ 1 + \sum_{x \in \text{GF}(q)} \sum_{y \in \text{GF}(q)} (x^2 + xy + ay^2) \\ \sum_{x \in \text{GF}(q)} \sum_{y \in \text{GF}(q)} 1 \end{pmatrix} = (0, 0, 1, 0)^T$$

and $\mathbf{G}_{\mathcal{T}}\mathbf{1}^T = (0, 0, 1, 0)^T$. For $S \in \{\mathcal{V}, \mathcal{T}\}$, it is easily verified that

$$\mathbf{G} = \begin{pmatrix} 0 \\ 0 \\ \mathbf{G}_S \\ -1 \\ 0 \end{pmatrix}$$

is a generator matrix of $\overline{\mathcal{C}_S}(-\mathbf{1})$. Note that $-(0, 0, -1, 0) \in \mathcal{S}$, we have $d(\overline{\mathcal{C}_S}(-\mathbf{1})^\perp) \leq 2$. It is clear that $d(\overline{\mathcal{C}_S}(-\mathbf{1})^\perp) \geq 2$. Hence, $d(\overline{\mathcal{C}_S}(-\mathbf{1})^\perp) = 2$. The weight distribution of $\overline{\mathcal{C}_S}(-\mathbf{1})$ follows from Theorem 4. This completes the proof. $\square$

For each $S \in \{\mathcal{V}, \mathcal{T}\}$, let

$$\mathbf{G}_S = \begin{pmatrix} \mathbf{s}_1^\top & \mathbf{s}_2^\top & \cdots & \mathbf{s}_{q^2+1}^\top \end{pmatrix},$$

then there are integer i and j such that $\mathbf{s}_i = (0, 0, 1, 0)$ and $\mathbf{s}_j = (0, 0, 0, 1)$. If $i = 1$ (resp. $j = 2$), let π_1 (resp. π_2) be the identity permutation on the coordinate set $\{1, 2, \dots, q^2 + 1\}$. If $i \neq 1$ (resp. $j \neq 2$), let π_1 (resp. π_2) be the transposition $(1, i)$ (resp. $(2, j)$) on the coordinate set $\{1, 2, \dots, q^2 + 1\}$. Let $\pi = \pi_1\pi_2$. Then π is a special permutation of $\{1, 2, \dots, q^2 + 1\}$ and $\pi^{-1} = \pi$. Define

$$\mathbf{P}_\pi = \begin{pmatrix} \mathbf{e}_{\pi(1)}^\top & \mathbf{e}_{\pi(2)}^\top & \cdots & \mathbf{e}_{\pi(q^2+1)}^\top \end{pmatrix},$$

where $\mathbf{e}_i$ denotes a row vector of length $q^2 + 1$ with 1 in the i -th position and 0 in every other position. Let

$$\mathbf{u} = (u_1, u_2, 1, \dots, 1)\mathbf{P}_\pi,$$

where $u_1 \in \text{GF}(q) \setminus \{0, -1\}$ and $u_2 \in \text{GF}(q) \setminus \{1\}$. The parameters of the extended code $\overline{\mathcal{C}_S}(\mathbf{u})$ are documented in the next theorem.

Lemma 16. ([4,9]) Let $q > 2$ be a prime power. Let C be a q -ary $[n, n - 4, 4]$ linear code. Then $n \leq q^2 + 1$.

Theorem 17. Let $q > 2$ be a prime power and $S \in \{\mathcal{V}, \mathcal{T}\}$. Then the nonstandardly extended code $\overline{\mathcal{C}_S}(\mathbf{u})$ has parameters $[q^2 + 2, 4, q^2 - q]$, and weight enumerator

$$1 + q^2(q - 1)z^{q^2 - q} + q(q - 1)(q^2 - q + 1)z^{q^2 - q + 1} + (q^2 - 1)z^{q^2} + q(q - 1)^2z^{q^2 + 1}.$$

The dual code $\overline{\mathcal{C}_S}(\mathbf{u})^\perp$ has parameters $[q^2 + 2, q^2 - 2, 3]$ and is distance-optimal.

Proof. It is easily verified that

$$\begin{aligned} \mathbf{G}_S \mathbf{u}^\top &= \mathbf{G}_S \mathbf{P}_\pi^\top (u_1, u_2, 1, \dots, 1)^\top \\ &= \mathbf{G}_S \mathbf{P}_{\pi^{-1}} (u_1, u_2, 1, \dots, 1)^\top \\ &= \mathbf{G}_S \mathbf{P}_\pi (u_1, u_2, 1, \dots, 1)^\top \\ &= u_1 \mathbf{s}_{\pi(1)} + u_2 \mathbf{s}_{\pi(2)} + \cdots + \mathbf{s}_{\pi(q^2+1)} \\ &= (0, 0, u_1, u_2 - 1)^\top. \end{aligned}$$

It follows that

$$\begin{pmatrix} 0 \\ 0 \\ \mathbf{G}_S \\ u_1 \\ u_2 - 1 \end{pmatrix}$$

is a generator matrix of $\overline{\mathcal{C}_S}(\mathbf{u})$. Since $d(\mathcal{C}_S^\perp) = 4$, we get that $\mathbf{s}_i \neq b\mathbf{s}_j$ for any $i \neq j$ and $b \in \text{GF}(q)^*$. It is easily checked that $(0, 0, u_1, u_2 - 1) \neq b\mathbf{s}_i$ for any $b \in \text{GF}(q)^*$ and $1 \leq i \leq q^2 + 1$. It follows that $d(\overline{\mathcal{C}_S}(\mathbf{u})^\perp) \geq 3$. Note that

$$(0, 0, u_1, u_2 - 1) = u_1(0, 0, 1, 0) + (u_2 - 1)(0, 0, 0, 1),$$

we have $d(\overline{\mathcal{C}_S}(\mathbf{u})^\perp) \leq 3$. Therefore, $d(\overline{\mathcal{C}_S}(\mathbf{u})^\perp) = 3$. By the sphere packing bound, a q -ary $[q^2 + 2, q^2 - 2, d \geq 5]$ linear code does not exist. By Lemma 16, a q -ary $[q^2 + 2, q^2 - 2, 4]$ linear code does not exist. Therefore, $\overline{\mathcal{C}_S}(\mathbf{u})^\perp$ is a distance-optimal code. The weight distribution of $\overline{\mathcal{C}_S}(\mathbf{u})$ follows from Theorem 4. This completes the proof. $\square$

For each $S \in \{\mathcal{V}, \mathcal{T}\}$, although the extended codes $\overline{\mathcal{C}_S(\mathbf{u})}$ in Theorem 17 and $\overline{\mathcal{C}_S(-\mathbf{1})}$ in Theorem 15 have the same parameters, $\overline{\mathcal{C}_S(\mathbf{u})}$ and $\overline{\mathcal{C}_S(-\mathbf{1})}$ are not equivalent, as they have different weight enumerators. The code $\overline{\mathcal{C}_S(\mathbf{u})}^\perp$ is a distance-optimal code and $d(\overline{\mathcal{C}_S(\mathbf{u})}^\perp) = 3$, while $d(\overline{\mathcal{C}_S(-\mathbf{1})}^\perp) = 2$. Furthermore, the standardly extended code of $\overline{\mathcal{C}_S(\mathbf{u})}$ has the following results.

Theorem 18. Let $S \in \{\mathcal{V}, \mathcal{T}\}$. Then the doubly extended code $\overline{\overline{\mathcal{C}_S(\mathbf{u})}}(-\mathbf{1})$ has parameters $[q^2 + 3, 4, q^2 - q]$. The dual code $\overline{\overline{\mathcal{C}_S(\mathbf{u})}}(-\mathbf{1})^\perp$ has parameters $[q^2 + 3, q^2 - 1, 3]$ and is distance-optimal.

Proof. Combining Theorems 15 and 17, we get

$$\begin{aligned}\overline{\mathbf{G}_S}(\mathbf{1}, \mathbf{1})^T &= \mathbf{G}_S \mathbf{1}^T + (0, 0, u_1, u_2 - 1)^T \\ &= (0, 0, 1, 0)^T + (0, 0, u_1, u_2 - 1)^T \\ &= (0, 0, u_1 + 1, u_2 - 1)^T,\end{aligned}$$

where $\mathbf{1}$ denotes the all-one vector over $\text{GF}(q)$ with length $q^2 + 1$. It follows that

$$\mathbf{G} = \begin{pmatrix} 0 & 0 \\ \mathbf{G}_S & 0 \\ u_1 & -(u_1 + 1) \\ u_2 - 1 & 1 - u_2 \end{pmatrix}$$

is a generator matrix of $\overline{\overline{\mathcal{C}_S(\mathbf{u})}}(-\mathbf{1})$. It is easily verified that $(\mathbf{c}, 0) \in \overline{\overline{\mathcal{C}_S(\mathbf{u})}}(-\mathbf{1})^\perp$ for any $\mathbf{c} \in \overline{\mathcal{C}_S(\mathbf{u})}$. By Theorem 17, $d(\overline{\mathcal{C}_S(\mathbf{u})}^\perp) = 3$. Then $d(\overline{\overline{\mathcal{C}_S(\mathbf{u})}}(-\mathbf{1})^\perp) \leq d(\overline{\mathcal{C}_S(\mathbf{u})}^\perp) = 3$. Similar to Theorem 17, we can prove that every two columns of $\mathbf{G}$ are linearly independent. Then $d(\overline{\overline{\mathcal{C}_S(\mathbf{u})}}(-\mathbf{1})^\perp) \geq 3$. Therefore, $d(\overline{\overline{\mathcal{C}_S(\mathbf{u})}}(-\mathbf{1})^\perp) = 3$. Similar to Theorem 17, we can prove that a q -ary $[q^2 + 3, q^2 - 1, d \geq 4]$ linear code does not exist. Therefore, $\overline{\overline{\mathcal{C}_S(\mathbf{u})}}(-\mathbf{1})^\perp$ is a distance-optimal code.

It is clear that $d(\overline{\overline{\mathcal{C}_S(\mathbf{u})}}(-\mathbf{1})) = q^2 - q$ or $q^2 - q + 1$. Consider the codeword $\mathbf{c} = (c_1, c_2, 0, 0)\mathbf{G} \in \overline{\overline{\mathcal{C}_S(\mathbf{u})}}(-\mathbf{1})$, where $(c_1, c_2) \neq (0, 0)$. It is easily verified that

$$\text{wt}(\mathbf{c}) = |\{(x, y) \in \text{GF}(q)^2 : c_1x + c_2y \neq 0\}| = q(q - 1).$$

Therefore, $d(\overline{\overline{\mathcal{C}_S(\mathbf{u})}}(-\mathbf{1})) = q^2 - q$. This completes the proof. $\square$

3.3.2. The standardly extended codes of a family of constacyclic ovoid codes

When q is odd, no cyclic code can be used to construct an ovoid in $\text{PG}(3, \text{GF}(q))$. But there is a λ -constacyclic construction of any ovoid in $\text{PG}(3, \text{GF}(q))$ for odd q due to the following theorem, which is known in the literature.

Theorem 19. Let $q > 2$ be a prime power, $m = 4$ and $h = q + 1$. Let $\mathcal{C}$ be the q -ary λ -constacyclic code in Theorem 6. Then $\mathcal{C}$ is a q -ary ovoid code, which is monomially equivalent to the elliptic quadric code.

Notice that the q in Theorem 19 could be even and odd, and the λ defined in Theorem 19 is a generator of $\text{GF}(q)^*$. Hence, the code $\mathcal{C}$ is neither cyclic nor negacyclic.

Theorem 20. Let $\mathcal{C}$ be the λ -constacyclic code in Theorem 19. Then $\overline{\mathcal{C}}(-\mathbf{1})$ has parameters $[q^2 + 2, 4, q^2 - q]$ and weight enumerator

$$1 + q^2(q - 1)z^{q^2 - q} + q(q - 1)(q^2 - q + 1)z^{q^2 - q + 1} + (q^2 - 1)z^{q^2} + q(q - 1)^2z^{q^2 + 1}.$$

The dual code $\overline{\mathcal{C}}(-\mathbf{1})^\perp$ has parameters $[q^2 + 2, q^2 - 2, 3]$ and is distance-optimal.

Proof. Let $\theta = \beta^{-1}$ and $\mathbb{M}_\theta(x)$ be the minimal polynomial of θ over $\text{GF}(q)$. Then $\mathcal{C}^\perp = \langle \mathbb{M}_\theta(x) \rangle$ is a q -ary λ^{-1} -constacyclic code of length n . We claim $\theta - 1 \notin \langle \theta \rangle$. Otherwise, suppose there are integers i and j such that $\theta - 1 = \theta^{ni+j}$, where $0 \leq i \leq q - 2$ and $0 \leq j \leq n - 1$. It follows that $1 - x + \theta^{ni}x^j \in \mathcal{C}^\perp$, which contradicts the fact that $d(\mathcal{C}^\perp) = 4$. By Lemma 8, $d(\overline{\mathcal{C}}(-\mathbf{1})^\perp) \geq 3$. Then the weight distribution of $\overline{\mathcal{C}}(-\mathbf{1})$ follows from Theorem 4.

By the fourth Pless power moment identity, we get

$$\overline{A}_3^\perp = \frac{q(q - 1)^2}{2}.$$

Therefore, $d(\overline{\mathcal{C}}(-\mathbf{1})^\perp) = 3$. This completes the proof. $\square$

3.3.3. Some nonstandardly extended codes of a family of cyclic ovoid codes

When q is even, there is a cyclic-code construction of the elliptic quadric [18, Chapter 13]. In this subsection, we study the nonstandardly extended codes of this family of cyclic ovoid codes.

Theorem 21. [18, Chapter 13] Let $q = 2^s$, where $s \geq 2$. Let β be a primitive $(q^2 + 1)$ -th root of unity in $\text{GF}(q^4)$ and $\mathbb{M}_\beta(x)$ be the minimal polynomial of β over $\text{GF}(q)$. Let $\mathcal{C}$ be the q -ary cyclic code of length n with check polynomial $\mathbb{M}_\beta(x)$. Then the code $\mathcal{C}$ has parameters $[q^2 + 1, 4, q^2 - q]$ and weight enumerator $1 + (q^2 - q)(q^2 + 1)z^{q^2 - q} + (q - 1)(q^2 + 1)z^{q^2}$.

Let $n = q^2 + 1$ and $\theta = \beta^{-1}$. Then $\mathcal{C}$ has the following trace representation:

$$\mathcal{C} = \left\{ (\text{Tr}_{q^4/q}(a), \text{Tr}_{q^4/q}(a\theta), \dots, \text{Tr}_{q^4/q}(a\theta^{n-1})) : a \in \text{GF}(q^4) \right\}.$$

It is clear that

$$\sum_{i=0}^{n-1} \text{Tr}_{q^4/q}(a\theta^i) = \text{Tr}_{q^4/q}\left(a \sum_{i=0}^{n-1} \theta^i\right) = 0.$$

Therefore, the standardly extended code $\overline{\mathcal{C}}(-\mathbf{1})$ is trivial. Below we shall study a nonstandardly extended code of $\mathcal{C}$.

Lemma 22. Let $\mathbf{u} = (1, u, \dots, u^{n-1})$, where $u \in \text{GF}(q) \setminus \{0, 1\}$. Let $\mathcal{C}$ be the cyclic code in Theorem 21. Then $d(\overline{\mathcal{C}}(\mathbf{u})^\perp) \geq 3$.

Proof. Let $\theta = \beta^{-1}$. Then $\mathcal{C}^\perp = (\mathbb{M}_\theta(x))$ and has parameters $[q^2 + 1, q^2 - 3, 4]$.

Suppose $\mathbf{u} \in \mathcal{C}^\perp$. Then $1 + u\theta + u^2\theta^2 + \dots + u^{n-1}\theta^{n-1} = 0$. It is clear that $u\theta \neq 1$. Consequently,

$$\sum_{i=0}^{n-1} (u\theta)^i = \frac{(u\theta)^n - 1}{u\theta - 1} = \frac{u^n - 1}{u\theta - 1} = \frac{u^2 - 1}{u\theta - 1} = 0,$$

which contradicts the fact that $u \notin \text{GF}(q) \setminus \{0, 1\}$. Therefore, $\mathbf{u} \notin \mathcal{C}^\perp$. By Lemma 2, $d(\overline{\mathcal{C}}(\mathbf{u})^\perp) \geq 2$.

Suppose $d(\overline{\mathcal{C}}(\mathbf{u})^\perp) = 2$. By Lemma 2, there are $a \in \text{GF}(q)^*$ and $1 \leq j \leq n$ such that $\mathbf{u} + a\mathbf{e}_j \in \mathcal{C}^\perp$. It is equivalent to

$$0 = \sum_{i=0}^{n-1} (u\theta)^i + a\theta^{j-1} = \frac{u^2 - 1}{u\theta - 1} + a\theta^{j-1}.$$

It follows that $au\theta^j - a\theta^{j-1} + u^2 - 1 = 0$, which contradicts the fact that $d(\mathcal{C}^\perp) = 4$. Therefore, $d(\overline{\mathcal{C}}(\mathbf{u})^\perp) \geq 3$. This completes the proof. $\square$

Theorem 23. Let $\mathbf{u} = (1, u, \dots, u^{n-1})$, where $u \in \text{GF}(q) \setminus \{0, 1\}$. Let $\mathcal{C}$ be the cyclic code in Theorem 21. Then $\overline{\mathcal{C}}(\mathbf{u})$ has parameters $[q^2 + 2, 4, q^2 - q]$ and weight enumerator

$$1 + q^2(q - 1)z^{q^2 - q} + q(q - 1)(q^2 - q + 1)z^{q^2 - q + 1} + (q^2 - 1)z^{q^2} + q(q - 1)^2z^{q^2 + 1}.$$

The dual code $\overline{\mathcal{C}}(\mathbf{u})^\perp$ has parameters $[q^2 + 2, q^2 - 2, 3]$ and is distance-optimal.

Proof. The proof of theorem is similar to the proof of Theorem 20, so it is omitted. $\square$

Although $\overline{\mathcal{C}}(\mathbf{u})$ and $\overline{\mathcal{C}}(-\mathbf{1})$ have the same parameters, $\overline{\mathcal{C}}(\mathbf{u})$ and $\overline{\mathcal{C}}(-\mathbf{1})$ are not equivalent. The code $\overline{\mathcal{C}}(\mathbf{u})^\perp$ is distance-optimal and has minimum distance 3, while $d(\overline{\mathcal{C}}(-\mathbf{1})^\perp) = 1$. Thus, some nonstandardly extended codes of some cyclic codes are more interesting.

3.4. The extended codes of the Denniston maximal arc codes

A maximal (n, h) -arc $\mathcal{A}$ in the projective plane $\text{PG}(2, \text{GF}(q))$ is a subset of $n = hq + h - q$ points such that every line meets $\mathcal{A}$ in 0 or h points. A line is called a *secant* if it meets $\mathcal{A}$, and *external line* otherwise. A line in $\text{PG}(2, \text{GF}(q))$ is said to be external to a maximal arc $\mathcal{A}$ if it has no intersection with $\mathcal{A}$. The set of lines external to $\mathcal{A}$ is a maximal (n', h') -arc $\mathcal{A}'$ in the dual plane, where $n' = h'q + h' - q$ and $h' = q/h$. $\mathcal{A}'$ is called the *dual* of $\mathcal{A}$.

Any point of $\text{PG}(2, \text{GF}(q))$ is a $(1, 1)$ -arc. The complement of any line is a maximal (q^2, q) -arc. These are called *trivial arcs*. It has been proved that only trivial maximal arcs exist in $\text{PG}(2, \text{GF}(q))$ for odd q . For even q , we have the following conclusion [26].

Theorem 24. Let $q > 2$ be even. A maximal (n, h) -arc $\mathcal{A}$ in $\text{PG}(2, \text{GF}(q))$ exists if and only if h divides q , where $2 \leq h < q$.

In 1969 Denniston employed a special set of conics to construct maximal arcs in the Desarguesian planes $\text{PG}(2, \text{GF}(q))$ for even q [12]. Below we introduce the Denniston arcs, and will assume that $q > 2$ is even.

Let $X^2 + \beta X + 1$ be an irreducible polynomial over $\text{GF}(q)$. It is known that the total number of such irreducible polynomials over $\text{GF}(q)$ is $q/2$ [26, Sec. 1.3]. Define

$$F_\lambda := \{(x, y, z) : \lambda x^2 + y^2 + \beta yz + z^2 = 0\}, \lambda \in \text{GF}(q) \cup \{\infty\}.$$

Since $X^2 + \beta X + 1$ is irreducible over $\text{GF}(q)$, F_0 has only the point $(1, 0, 0)$, and F_∞ is the line $x = 0$. Every other conic F_λ is non-degenerate and has nucleus F_0 . Further, this pencil of conics $\{F_\lambda : \lambda \in \text{GF}(q) \cup \{\infty\}\}$ is a partition of the projective plane.

Let $\lambda \in \text{GF}(q)^*$. Then $\lambda x^2 + y^2 + \beta yz + z^2 = 0$ if and only if $x = \lambda^{-(q/2)}[y + (\beta yz)^{q/2} + z]$. It then follows that

$$F_\lambda := \left\{ \left(\lambda^{-(q/2)}, 1, 0 \right) \right\} \cup \left\{ \left(\lambda^{-(q/2)} \left(y + (\beta y)^{q/2} + 1 \right), y, 1 \right) : y \in \text{GF}(q) \right\}$$

for all $\lambda \in \text{GF}(q)^*$.

The following theorem documents *Denniston arcs* [12].

Theorem 25. *Let A be an additive subgroup of $\text{GF}(q)$ of order h . Then the set of points of all conics F_λ for $\lambda \in A$ form a maximal (n, h) -arc $\mathcal{A}$, where $n = hq + h - q$.*

More families of maximal arcs were discovered and reported in [50], [60] and [61]. Their relations with the Denniston arcs were discussed in [23].

Let $A = \{0, \lambda_1, \dots, \lambda_{h-1}\}$ be an additive subgroup of $\text{GF}(q)$ and $\text{GF}(q) = \{x_1, x_2, \dots, x_q\}$. Define

$$\mathbf{G}_{\lambda_i} = \begin{pmatrix} \lambda_i^{-(q/2)} & \lambda_i^{-(q/2)} f(x_1) & \lambda_i^{-(q/2)} f(x_2) & \cdots & \lambda_i^{-(q/2)} f(x_q) \\ 1 & x_1 & x_2 & \cdots & x_q \\ 0 & 1 & 1 & \cdots & 1 \end{pmatrix},$$

where $f(x) = x + (\beta x)^{q/2} + 1$. Denote

$$\mathbf{G}_{\mathcal{A}} = \begin{pmatrix} 1 \\ 0 \ \mathbf{G}_{\lambda_1} \ \mathbf{G}_{\lambda_2} \ \cdots \ \mathbf{G}_{\lambda_{h-1}} \\ 0 \end{pmatrix}.$$

We use $\mathcal{C}_{\mathcal{A}}$ to denote the q -ary code spanned by the rows of the matrix $\mathbf{G}_{\mathcal{A}}$, which is called a *maximal arc code*. A proof of the following theorem can be found in [18, Chapter 12].

Theorem 26. *Let $q = 2^m$ for any $m \geq 2$ and $h = 2^i$ with $1 \leq i < m$. Let $\mathcal{A}$ be a maximal (n, h) -arc in $\text{PG}(2, \text{GF}(q))$. Then the arc code $\mathcal{C}_{\mathcal{A}}$ has parameters $[n, 3, n - h]$ and weight enumerator*

$$1 + \frac{(q^2 - 1)n}{h} z^{n-h} + \frac{(q^3 - 1)h - (q^2 - 1)n}{h} z^n, \quad (24)$$

where $n = hq + h - q$. Further, $d^\perp = 4$ if $h = 2$ and $d^\perp = 3$ otherwise.

The parameters of the extended code of the q -ary arc code $\mathcal{C}_{\mathcal{A}}$ are documented in the next theorem.

Theorem 27. *Let $q = 2^m$ for any $m \geq 2$ and $h = 2^i$ with $1 \leq i < m$. Let $\mathcal{A}$ be a maximal (n, h) -arc in $\text{PG}(2, \text{GF}(q))$. Let $\mathbf{u} \in \Pi(\mathcal{C}_{\mathcal{A}})$. Then $\overline{\mathcal{C}_{\mathcal{A}}}(\mathbf{u})$ has parameters $[hq + h - q + 1, 3, hq - q]$.*

Proof. Recall that $\mathcal{C}_{\mathcal{A}}$ has parameters $[n, 3, n - h]$, where $n = hq + h - q$. It is clear that $\overline{\mathcal{C}_{\mathcal{A}}}(\mathbf{u})$ has length $n + 1$, dimension 3 and $d(\overline{\mathcal{C}_{\mathcal{A}}}(\mathbf{u})) = d(\mathcal{C}_{\mathcal{A}})$ or $d(\overline{\mathcal{C}_{\mathcal{A}}}(\mathbf{u})) = d(\mathcal{C}_{\mathcal{A}}) + 1$. Let $w_1 = n - h$ and $w_2 = n$. It follows from Theorem 4 that

$$\overline{A}_{w_1} = \frac{(q^2 - 1)w_2 - q(q - 1)n + q\overline{A}_2^\perp}{h} = \frac{(q - 1)n + q\overline{A}_2^\perp}{h} > 0,$$

where $\overline{A}_2^\perp \in \{0, q - 1\}$ and $\overline{A}_{w_1}$ denotes the total number of codewords with weight i in $\overline{\mathcal{C}_{\mathcal{A}}}(\mathbf{u})$. Therefore, $d(\overline{\mathcal{C}_{\mathcal{A}}}(\mathbf{u})) = d(\mathcal{C}_{\mathcal{A}})$. This completes the proof. $\square$

In the rest of this subsection, we use $\mathcal{A}$ to denote the Denniston maximal arc and study the standardly extended code $\overline{\mathcal{C}_{\mathcal{A}}}(-1)$ of the Denniston maximal arc code $\mathcal{C}_{\mathcal{A}}$.

Theorem 28. Let $q = 2^m$ with $m \geq 2$ and $h = 2$. Then the code $\overline{\mathcal{C}_{\mathcal{A}}}(-\mathbf{1})$ has weight enumerator

$$1 + \frac{q^2 + q - 2}{2}z^q + \frac{q^3 + q^2 - 2q}{2}z^{q+1} + \frac{q^2 - q}{2}z^{q+2} + \frac{q^3 - 3q^2 + 2q}{2}z^{q+3}.$$

The dual code $\overline{\mathcal{C}}(-\mathbf{1})^\perp$ has parameters $[q + 3, q, 3]$ and is distance-optimal. Hence, $\overline{\mathcal{C}_{\mathcal{A}}}(-\mathbf{1})$ is a $[q + 3, 3, q]$ NMDS code.

Proof. Let $A = \{0, \lambda\}$, where $\lambda \in \text{GF}(q)^*$. It is easily checked that $\mathbf{G}_{\mathcal{A}}\mathbf{1}^T = (1 + \lambda^{-(q/2)}, 1, 0)^T$. Then

$$\mathbf{G}_{\mathcal{A}}^{(1)} := \begin{pmatrix} 1 + \lambda^{-(q/2)} \\ \mathbf{G}_{\mathcal{A}} & 1 \\ 0 \end{pmatrix}$$

is a generator matrix of $\overline{\mathcal{C}_{\mathcal{A}}}(-\mathbf{1})$. It is easily seen that any two column vectors of $\mathbf{G}_{\mathcal{A}}^{(1)}$ are not linearly dependent. Hence, $d(\overline{\mathcal{C}_{\mathcal{A}}}(-\mathbf{1})^\perp) \geq 3$. Note that $(1 + \lambda^{-(q/2)}, 1, 0) = (1, 0, 0) + (\lambda^{-(q/2)}, 1, 0)$, it follows that $d(\overline{\mathcal{C}_{\mathcal{A}}}(-\mathbf{1})^\perp) \leq 3$. Therefore, $d(\overline{\mathcal{C}_{\mathcal{A}}}(-\mathbf{1})^\perp) = 3$. It follows from the definition of the second row of the generator matrix $\mathbf{G}_{\mathcal{A}}$ that $\mathbf{1} \notin \mathcal{C}_{\mathcal{A}}^\perp$. Consequently, $\mathbf{1} \in \Pi(\mathcal{C}_{\mathcal{A}})$. The weight distribution of $\overline{\mathcal{C}_{\mathcal{A}}}(-\mathbf{1})$ then follows from Theorem 4.

Suppose there is a q -ary $[q + 3, q, d \geq 4]$ linear code. By [30, Theorem 1.5.1], there exists a q -ary $[q + 2, q, d \geq 3]$ linear code. By the sphere packing bound, we get $1 + (q + 2)(q - 1) \leq q^2$, a contradiction. Therefore, $\overline{\mathcal{C}_{\mathcal{A}}}(-\mathbf{1})^\perp$ is a distance-optimal code. This completes the proof. $\square$

Theorem 29. Let $q = 2^m$ with $m \geq 2$ and $h = 2^i$, where $i \mid m$ and $2 \leq i < m$. Let A be a subfield of $\text{GF}(q)$ with h elements and let $\mathcal{A} = \{F_\lambda : \lambda \in A\}$. Then $\overline{\mathcal{C}_{\mathcal{A}}}(-\mathbf{1})$ has weight enumerator

$$1 + (q^2 - 1)z^{n-h} + \frac{(q^2 - 1)q(h - 1)}{h}z^{n-h+1} + \frac{(q - 1)(q^2 + q - qh)}{h}z^{n+1},$$

where $n = hq + h - q$. The dual code $\overline{\mathcal{C}}(-\mathbf{1})^\perp$ has parameters $[n + 1, n - 2, 2]$.

Proof. Let $A^* = A \setminus \{0\}$, then $|A^*| = h - 1$. It is easily verified that

$$\mathbf{G}_{\mathcal{A}}\mathbf{1}^T = \begin{pmatrix} 1 + \sum_{\lambda \in A^*} [\lambda^{-(q/2)} + \sum_{y \in \text{GF}(q)} \lambda^{-(q/2)}(y + (\beta y)^{q/2} + 1)] \\ \sum_{\lambda \in A^*} [1 + \sum_{y \in \text{GF}(q)} y] \\ \sum_{\lambda \in A^*} \sum_{y \in \text{GF}(q)} 1 \end{pmatrix}.$$

It is clear that $\sum_{y \in \text{GF}(q)} y = 0$. Consequently,

$$\begin{aligned} & \sum_{y \in \text{GF}(q)} \lambda^{-(q/2)}(y + (\beta y)^{q/2} + 1) \\ &= \lambda^{-(q/2)} \sum_{y \in \text{GF}(q)} y + \lambda^{-(q/2)} \beta^{q/2} \left(\sum_{y \in \text{GF}(q)} y \right)^{q/2} + \lambda^{-(q/2)} \sum_{y \in \text{GF}(q)} 1 = 0. \end{aligned}$$

Since A is a subfield of $\text{GF}(q)$, it follows that

$$\sum_{\lambda \in A^*} \lambda^{-(q/2)} = \left(\sum_{\lambda \in A^*} \lambda \right)^{q/2} = 0.$$

Therefore, $\mathbf{G}_{\mathcal{A}}\mathbf{1}^T = (1, 1, 0)^T$. It then follows that

$$\begin{pmatrix} 1 \\ \mathbf{G}_{\mathcal{A}} & 1 \\ 0 \end{pmatrix}$$

is a generator matrix of $\overline{\mathcal{C}_{\mathcal{A}}}(-\mathbf{1})$. Note that $1 \in A$, we have $(1, 1, 0) \in \mathcal{A}$. It follows that $d(\overline{\mathcal{C}_{\mathcal{A}}}(-\mathbf{1})^\perp) \leq 2$. Clearly, $d(\overline{\mathcal{C}_{\mathcal{A}}}(-\mathbf{1})^\perp) \geq 2$. Consequently, $d(\overline{\mathcal{C}_{\mathcal{A}}}(-\mathbf{1})^\perp) = 2$. It follows from the definition of the second row of the generator matrix $\mathbf{G}_{\mathcal{A}}$ that $\mathbf{1} \notin \mathcal{C}_{\mathcal{A}}^\perp$. Consequently, $\mathbf{1} \in \Pi(\mathcal{C}_{\mathcal{A}})$. The weight distribution of $\overline{\mathcal{C}_{\mathcal{A}}}(-\mathbf{1})$ then follows from Theorem 4. This completes the proof. $\square$

Now we consider a nonstandardly extended code of the Deniston arc code $\mathcal{C}_{\mathcal{A}}$.

Lemma 30. Let $q = 2^m$ for any $m \geq 2$ and $h = 2^i$, where $2 \leq i < m$. Let A be an additive subgroup of $\text{GF}(q)$ of order h . Then $\sum_{a \in A} a = 0$.

Proof. It is clear that any additive subgroup A of $\text{GF}(q)$ is a vector space over $\text{GF}(2)$. Consider the polynomial

$$L_A(x) = \prod_{a \in A} (x - a),$$

which is a 2-polynomial over $\text{GF}(2)$ (see [37, Theorem 3.52]). That is to say,

$$L_A(x) = x^{2^i} + \sum_{j=0}^{i-1} a_j x^{2^j},$$

for some $a_j \in \text{GF}(2)$. It is clear that $\sum_{a \in A} a$ is the coefficient of x^{2^i-1} in the expansion of $L_A(x)$. Therefore, $\sum_{a \in A} a = 0$. This completes the proof. $\square$

Theorem 31. Let $q = 2^m$ for any $m \geq 2$ and $h = 2^i$, where $2 \leq i < m$. Let $\mathcal{A}$ be a maximal (n, h) -arc in $\text{PG}(2, \text{GF}(q))$. Let $\mathbf{u} = (0, \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{h-1})$, where $\mathbf{u}_1 = (0, 1, \dots, 1) \in \text{GF}(q)^{q+1}$ and $\mathbf{u}_i = (\lambda_i^{q/2}, 1, \dots, 1) \in \text{GF}(q)^{q+1}$ for $2 \leq i \leq h-1$. Then $\overline{\mathcal{C}}_{\mathcal{A}}(\mathbf{u})$ has weight enumerator

$$1 + \frac{q^2 h - q^2 + q - h}{h} z^{n-h} + \frac{q^3 h - q^3 + q^2 - qh}{h} z^{n-h+1} + \frac{q^2 - q}{h} z^n + \frac{q^3 - q^2 h - q^2 + qh}{h} z^{n+1},$$

where $n = hq + h - q$. The dual code $\overline{\mathcal{C}}_{\mathcal{A}}(\mathbf{u})^\perp$ has parameters $[n+1, n-2, 3]$ and is distance-optimal.

Proof. It is easy to check that $\mathbf{G}_{\lambda_1} \mathbf{u}_1^T = (0, 0, 0)^T$ and $\mathbf{G}_{\lambda_i} \mathbf{u}_i^T = (1, \lambda_i^{q/2}, 0)^T$ for each $2 \leq i \leq h-1$. Therefore,

$$\mathbf{G}_{\mathcal{A}} \mathbf{u}^T = \sum_{i=1}^{h-1} \mathbf{G}_{\lambda_i} \mathbf{u}_i^T = \left(\sum_{i=2}^{h-1} 1, \sum_{i=2}^{h-1} \lambda_i^{q/2}, 0 \right)^T.$$

It follows from Lemma 30 that $\sum_{i=1}^{h-1} \lambda_i = 0$. Hence, $\sum_{i=2}^{h-1} \lambda_i^{q/2} = \lambda_1^{q/2}$. It then follows that

$$\mathbf{G}_{\mathcal{A}} \mathbf{u}^T = (0, \lambda_1^{q/2}, 0)^T.$$

Consequently,

$$\mathbf{G}_{\mathcal{A}}^{(1)} := \begin{pmatrix} 0 \\ \mathbf{G}_{\mathcal{A}} \lambda_1^{q/2} \\ 0 \end{pmatrix}$$

is a generator matrix of $\overline{\mathcal{C}}_{\mathcal{A}}(\mathbf{u})$. It is easily verified that any two column vectors of $\mathbf{G}_{\mathcal{A}}^{(1)}$ are not proportional. Hence, $d(\overline{\mathcal{C}}_{\mathcal{A}}(\mathbf{u})^\perp) \geq 3$. Note that $d(\overline{\mathcal{C}}_{\mathcal{A}}(\mathbf{u})^\perp) \leq d(\mathcal{C}_{\mathcal{A}}^\perp) = 3$. Therefore, $d(\overline{\mathcal{C}}_{\mathcal{A}}(\mathbf{u})^\perp) = 3$. The weight distribution of $\overline{\mathcal{C}}_{\mathcal{A}}(\mathbf{u})$ then follows from Theorem 4.

Suppose there is a q -ary $[n+1, n-2, d \geq 4]$ linear code. By [30, Theorem 1.5.1], there exists a q -ary $[n, n-2, d \geq 3]$ linear code. By the sphere packing bound, we get $1 + n(q-1) \leq q^2$, which contradicts the fact that $n \geq 3q+4$. Therefore, $\overline{\mathcal{C}}_{\mathcal{A}}(\mathbf{u})^\perp$ is a distance-optimal code. This completes the proof. $\square$

4. A summary of the contributions of this paper

In this paper, a fundamental result about the extended codes of the projective two-weight codes was derived in Theorem 4, and the parameters of the extended codes of several families of the projective two-weight codes were presented. A summary of the major specific contributions of this paper goes as follows.

- The weight distribution of a family of constacyclic projective two-weight codes was determined in Subsection 3.2.
- The extended codes of some ovoid codes $\mathcal{C}$ were studied in Subsection 3.3, and the following main results were obtained:
 - The extended code $\overline{\mathcal{C}}^\perp(\mathbf{u})$ is a q -ary $[q^2+2, q^2-3, 4]$ linear code and is distance-optimal (see Theorem 14).
 - The standardly extended code $\overline{\mathcal{C}}(-\mathbf{1})$ is a q -ary $[q^2+2, 4, q^2-q]$ linear code and $d(\overline{\mathcal{C}}(-\mathbf{1})^\perp) = 2$ (see Theorem 15). Further, there is a vector $\mathbf{u}$ such that the following hold:
 - * The nonstandardly extended code $\overline{\mathcal{C}}(\mathbf{u})$ is a q -ary $[q^2+2, 4, q^2-q]$ linear code. The dual code $\overline{\mathcal{C}}(\mathbf{u})^\perp$ is a q -ary $[q^2+2, q^2-2, 3]$ linear code and is distance-optimal (see Theorem 17).

- * The doubly extended code $\overline{\overline{\mathcal{C}(\mathbf{u})}}(-\mathbf{1})$ is a q -ary $[q^2 + 3, 4, q^2 - q]$ linear code. The dual code $\overline{\overline{\mathcal{C}(\mathbf{u})}}(-\mathbf{1})^\perp$ is a q -ary $[q^2 + 3, q^2 - 1, 3]$ linear code and is distance-optimal (see Theorem 18).
- The standardly extended codes of a family of constacyclic ovoid codes $\mathcal{C}$ were studied in Subsection 3.3. The code $\overline{\mathcal{C}}(-\mathbf{1})$ is a q -ary $[q^2 + 2, 4, q^2 - q]$ linear code. The dual code $\overline{\mathcal{C}}(-\mathbf{1})^\perp$ is a q -ary $[q^2 + 2, q^2 - 2, 3]$ linear code and is distance-optimal (see Theorem 20).
- The extended codes of a family of cyclic ovoid codes $\mathcal{C}$ were studied in Subsection 3.3, and the following main results were obtained:
 - The standardly extended code $\overline{\mathcal{C}}(-\mathbf{1})$ is trivial.
 - There is a vector $\mathbf{u}$ such that $\overline{\mathcal{C}}(\mathbf{u})$ is a q -ary $[q^2 + 2, 4, q^2 - q]$ linear code. The code $\overline{\mathcal{C}}(\mathbf{u})^\perp$ is a q -ary $[q^2 + 2, q^2 - 2, 3]$ linear code and is distance-optimal (see Theorem 23).
- The extended codes of the Denniston maximal arc codes $\mathcal{C}$ were studied in Subsection 3.4, and the following main results were obtained:
 - The extended code $\overline{\mathcal{C}}(\mathbf{u})$ is a q -ary $[hq + h - q + 1, 3, hq - q]$ linear code, where $q = 2^m$ and $h = 2^i$ with $1 \leq i < m$ (see Theorem 27).
 - When $h = 2$, the dual code $\overline{\mathcal{C}}(-\mathbf{1})^\perp$ is a q -ary $[q + 3, q, 3]$ linear code and is distance-optimal (see Theorem 28).
 - When $i \mid m$ and $i \geq 2$, the dual code $\overline{\mathcal{C}}(-\mathbf{1})^\perp$ has minimum distance 2 (see Theorem 29).
 - There is a vector $\mathbf{u}$ such that $\overline{\mathcal{C}}(\mathbf{u})^\perp$ is a q -ary $[hq + h - q + 1, hq + h - q - 2, 3]$ AMDS code for $h \geq 4$ and even q (see Theorem 31).

In summary, six infinite families of distance-optimal were obtained in this paper via the extending technique.

CRedit authorship contribution statement

The conceptualization, methodology, investigation, writing-original draft preparation, review and revision-editing have been performed by three the authors equally.

Declaration of competing interest

All the authors declare they have no conflicts of interest.

Data availability

No data was used for the research described in the article.

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