Solving Recurrence Relations

Cunsheng Ding

HKUST, Hong Kong

October 10, 2015
Contents

1. Introduction
2. Linear Recurrence Relations
3. Solving Linear Homogeneous Recurrence Relations
4. Generating Functions and Linear Recursions
5. Solving Nonlinear Recurrence Relations
Recursions and linear recursions were introduced in the previous lecture. The objectives of this lecture are the following.

- Recall the definitions of linear recurrence relations.
- Introduce general techniques for solving linear recurrence relations.
- Solving a number of important types of linear recurrence relations.
- Solving nonlinear recurrence relations.

These techniques will be fundamental in the design and analysis of computer algorithms.
Definition 1

A linear recurrence relation with constant coefficients for a sequence \((s_i)_{i=0}^{\infty}\) is a formula that relates each term \(s_i\) to its predecessors \(s_{i-1}, s_{i-2}, \ldots, s_{i-\ell}\) in the form

\[s_i = c_1 s_{i-1} + c_2 s_{i-2} + \cdots + c_\ell s_{i-\ell} + d \quad \text{for all } i \geq \ell,\]

where \(\ell\) is some fixed integer and \(d\) is a constant.

Example 2

Let \((s_i)_{i=0}^{\infty}\) be defined by \(s_i = i\) for all integers \(i \geq 0\). Then \(s_i = s_{i-1} + 1\) is a linear recurrence relation for the sequence with the initial condition that \(s_0 = 0\).
Linear Homogeneous Recurrence Relations

Definition 3

A linear homogeneous recurrence relation of degree \( \ell \) with constant coefficients (in sort, LHRRCC) for a sequence \((s_i)_{i=0}^{\infty}\) is a formula that relates each term \(s_i\) to its predecessors \(s_{i-1}, s_{i-2}, \ldots, s_{i-\ell}\) in the form

\[
s_i = c_1 s_{i-1} + c_2 s_{i-2} + \cdots + c_\ell s_{i-\ell} \text{ for all } i \geq \ell,
\]

where \(\ell\) is some fixed integer, and \(c_i\)'s are real constants with \(c_\ell \neq 0\). The equation

\[
x^\ell - c_1 x^{\ell-1} - c_2 x^{\ell-2} - \cdots - c_{\ell-1} x - c_\ell = 0
\]

is called the characteristic equation of the linear recursion of (2), and its roots are referred to as the characteristic roots. The polynomial \(x^\ell - c_1 x^{\ell-1} - c_2 x^{\ell-2} - \cdots - c_{\ell-1} x - c_\ell\) is called the characteristic polynomial of the sequence.
Question 1

Given a sequence \((s_i)_{i=i_0}^{\infty}\) defined by a linear homogeneous recurrence relation with constant coefficients, how do you solve the LHRCC so that you are able to find a mathematical formula for each term of the sequence?

Example 4

Let \((s_i)_{i=0}^{\infty}\) be defined by the following linear homogeneous recurrence relation of degree 2:

\[ s_{i+1} = 2s_i - s_{i-1} \text{ for all } i \geq 1 \]

with initial conditions \(s_0 = 1\) and \(s_1 = 3\). Find a mathematical formula in terms of \(i\) for each \(s_i\).
When the Characteristic Roots Have Multiplicity 1

**Recurrence:** \( s_i = c_1 s_{i-1} + c_2 s_{i-2} + \cdots + c_\ell s_{i-\ell} \) for all \( i \geq \ell \).

**Characteristic equation:** \( x^\ell - c_1 x^{\ell-1} - c_2 x^{\ell-2} - \cdots - c_{\ell-1} x - c_\ell = 0 \).

**Theorem 5**

*If the characteristic equation has distinct roots \( r_1, r_2, \ldots, r_\ell \), then a sequence \((s_i)_{i=0}^{\infty}\) satisfies the linear recurrence relation if and only if*

\[
s_i = \alpha_1 r_1^i + \alpha_2 r_2^i + \cdots + \alpha_\ell r_\ell^i \quad \text{for integers } i \geq 0, \tag{4}
\]

*where \( \alpha_1, \alpha_2, \ldots, \alpha_\ell \) are constants.*

**Remarks**

A proof of the necessity will be presented in a tutorial. The proof of the sufficiency will be left as an assignment problem.
When the Characteristic Roots Have Multiplicity 1

Recurrence: \( s_i = c_1 s_{i-1} + c_2 s_{i-2} + \cdots + c_\ell s_{i-\ell} \) for all \( i \geq \ell \).

Characteristic equation: \( x^\ell - c_1 x^{\ell-1} - c_2 x^{\ell-2} - \cdots - c_{\ell-1} x - c_\ell = 0 \).

Steps in solving the recurrence relation

1. Solving the characteristic equation to find out all the distinct roots \( r_1, r_2, \ldots, r_\ell \).

2. Use the initial conditions \( s_0, s_1, \ldots, s_{\ell-1} \) and the roots \( r_i \) to solve the following set of equations,

\[
s_i = \alpha_1 r_1^i + \alpha_2 r_2^i + \cdots + \alpha_\ell r_\ell^i, \quad i = 0, 1, 2, \ldots, \ell - 1.
\]

This will determine \( \alpha_1, \alpha_2, \ldots, \alpha_\ell \).
Solving the First-order Linear Homogeneous Recurrence Relations

**Recurrence:** \( s_i = c_1 s_{i-1} \) for all \( i \geq 1 \).

**Characteristic equation:** \( x - c_1 = 0 \).

**Steps in solving the recurrence relation**

1. Solving the characteristic equation to find out the unique root \( r_1 = c_1 \).
2. Use the initial condition \( s_0 \) and the root \( r_1 \) to solve the following equation

\[
    s_0 = \alpha_1.
\]

This will determine \( \alpha_1 = s_0 \).

Hence, \( s_i = s_0 c_1^i \) for all integers \( i \geq 0 \). This is the geometric sequence.
Solving the Second-order Linear Homogeneous Recurrence Relations

Recurrence: \( s_i = c_1 s_{i-1} + c_2 s_{i-2} \) for all \( i \geq 2 \).

Characteristic equation: \( x^2 - c_1 x - c_2 = 0 \).

Steps in solving the recurrence relation

1. Solving the characteristic equation to find out the two distinct roots \( r_1, r_2 \).
2. Use the initial conditions \( s_0, s_1 \) and the roots \( r_1, r_2 \) to solve the following set of equations,

\[
\begin{align*}
    s_0 &= \alpha_1 + \alpha_2, \\
    s_1 &= \alpha_1 r_1 + \alpha_2 r_2.
\end{align*}
\]

This yields \( \alpha_1 \) and \( \alpha_2 \).

By Theorem 5, we have

\[
s_i = \frac{s_1 - s_0 r_2}{r_1 - r_2} r_1^i + \frac{s_0 r_1 - s_1}{r_1 - r_2} r_2^i.
\]
The Fibonacci Sequence

Problem 6

The sequence \((F_i)_{i=0}^{\infty}\) is defined by the linear homogeneous recursion

\[ F_i = F_{i-1} + F_{i-2} \text{ for all } i \geq 2, \]

with initial condition \(F_0 = 0\) and \(F_1 = 1\). Solve this linear recurrence relation.

Solution 7

The characteristic equation \(x^2 - x - 1 = 0\) has the following distinct roots

\[ r_1 = \frac{1 + \sqrt{5}}{2}, \quad r_2 = \frac{1 - \sqrt{5}}{2}. \]

Hence,

\[ F_i = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^i - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^i \]
Problem 8

Solve the following linear recurrence relation

\[ s_i = 6s_{i-1} - 11s_{i-2} + 6s_{i-3} \text{ for all } i \geq 3 \]

with initial conditions \( s_0 = 2 \), \( s_1 = 5 \) and \( s_2 = 15 \).
When the Characteristic Roots Have Multiplicity $> 1$

Recurrence: $s_i = c_1 s_{i-1} + c_2 s_{i-2} + \cdots + c_\ell s_{i-\ell}$ for all $i \geq \ell$.

Characteristic equation: $x^\ell - c_1 x^{\ell-1} - c_2 x^{\ell-2} - \cdots - c_{\ell-1} x - c_\ell = 0$.

**Theorem 9**

If the characteristic equation has distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$, respectively, so that all $m_i$’s are positive and $\sum_{i=1}^t m_i = \ell$, then a sequence $(s_i)_{i=0}^\infty$ satisfies the linear recurrence relation if and only if

$$s_i = (\alpha_{1,0} + \alpha_{1,1} i + \cdots + \alpha_{1,m_1-1} i^{m_1-1}) r_1^i + (\alpha_{2,0} + \alpha_{2,1} i + \cdots + \alpha_{2,m_2-1} i^{m_2-1}) r_2^i + \cdots + (\alpha_{t,0} + \alpha_{t,1} i + \cdots + \alpha_{t,m_t-1} i^{m_t-1}) r_t^i \text{ for all } i \geq 0,$$

where all $\alpha_{i,j}$’s are constants.
When the Characteristic Roots Have Multiplicity > 1

Recurrence: \( s_i = c_1 s_{i-1} + c_2 s_{i-2} + \cdots + c_\ell s_{i-\ell} \) for all \( i \geq \ell \).

Characteristic equation: \( x^\ell - c_1 x^{\ell-1} - c_2 x^{\ell-2} - \cdots - c_{\ell-1} x - c_\ell = 0 \).

Steps in solving the recurrence relation

1. Solving the characteristic equation to find out all the distinct roots \( r_1, r_2, \ldots, r_t \) and their multiplicities.
2. Use the initial conditions \( s_0, s_1, \ldots, s_{\ell-1} \) and the roots \( r_i \)'s and their multiplicities \( m_i \) to solve the following set of equations,

   \[
   s_i = (\alpha_{1,0} + \alpha_{1,1} i + \cdots + \alpha_{1,m_1-1} i^{m_1-1}) r_1^i + \\
   = (\alpha_{2,0} + \alpha_{2,1} i + \cdots + \alpha_{2,m_2-1} i^{m_2-1}) r_2^i + \cdots + \\
   = (\alpha_{t,0} + \alpha_{t,1} i + \cdots + \alpha_{t,m_t-1} i^{m_t-1}) r_t^i, \quad i = 0, 1, \ldots, \ell - 1.
   \]

   This will determine \( \alpha_{i,j} \)'s.

Remark: We will not present a proof for this theorem.
When the Characteristic Roots Have Multiplicity \( > 1 \)

**Recurrence:** \( s_i = 6s_{i-1} - 9s_{i-2} \) for all \( i \geq 2 \) with \( s_0 = 1 \) and \( s_1 = 6 \).

**Characteristic equation:** \( x^2 - 6x + 9 = 0 \).

**Solution 10**

Note that \( x^2 - 6x + 9 = 0 \) has the only root \( x = 3 \) with multiplicity 2. By Theorem 9,

\[
 s_i = \alpha_1 3^i + \alpha_2 i3^i.
\]

Using the initial conditions, we obtain that \( \alpha_1 = \alpha_2 = 1 \). Hence,

\[
 s_i = (i + 1)3^i.
\]
A rational function is the quotient of two “polynomials” of finite degree over the set of real numbers.

Example 12

\[
\frac{x + x^2}{1 - 3x + 3x^2 - x^3}.
\]
Sequences Defined by Rational Functions

Theorem 13 (Power series expansion of a rational function)

Every rational function \( f(x)/g(x) \) can be expressed as

\[
\frac{f(x)}{g(x)} = \sum_{i=0}^{\infty} s_i x^i
\]

where \( \gcd(f(x), g(x)) = 1 \), \( \deg(f) < \deg(g) \) and \( g(0) \neq 0 \).

Proof.

Let

\[
f(x) = g(x) \sum_{i=0}^{\infty} s_i x^i.
\]

Solving the polynomial equation above yields \( s_i \) one by one.

Remark: \( (s_i)_{i=0}^{\infty} \) is the sequence defined by the rational function \( f(x)/g(x) \).
Sequences Defined by Rational Functions

Example 14

\[
\frac{1}{1 - x} = \sum_{k=0}^{\infty} x^k.
\]

The sequence is \((1)_{i=0}^{\infty}\).

Example 15

\[
\frac{1}{1 - 2x} = \sum_{k=0}^{\infty} 2^k x^k.
\]

The sequence is \((2^i)_{i=0}^{\infty}\).
Definition 16

The **generating function** of an infinite sequence \( (s_i)_{i=0}^{\infty} \) is defined by

\[
S(x) = \sum_{i=0}^{\infty} s_i x^i.
\]

Example 17

The generating function of the constant sequence \( (1)_{i=0}^{\infty} \) is defined by

\[
S(x) = \sum_{i=0}^{\infty} x^i = \frac{1}{1-x}.
\]
Definition

The **generating function** of an infinite sequence \((s_i)_{i=0}^{\infty}\) is defined by

\[ S(x) = \sum_{i=0}^{\infty} s_i x^i. \]

Questions

- Can the generating function of a sequence always be expressed as a rational function?
- If the answer is Yes, please give a proof.
- If the answer is No, please give a counter-example and derive conditions under which the generating function of a sequence can be expressed as a rational function.
Generating Functions and Linear Recursions

Example 18

Let \((s_i)_{i=0}^{\infty}\) be a sequence defined by \(s_i = 5s_{i-1} - 6s_{i-2}, \quad i \geq 2\), with initial condition \(s_0 = 1\) and \(s_1 = -2\). Employing this linear recurrence relation,

\[
S(x) = s_0 + s_1 x + s_2 x^2 + s_3 x^3 + s_4 x^4 + \cdots
\]

\[-5xS(x) = -5s_0 x - 5s_1 x^2 - 5s_2 x^3 - 5s_3 x^4 - \cdots
\]

\[6x^2 S(x) = + 6s_0 x^2 + 6s_1 x^3 + 6s_2 x^4 + \cdots
\]

Hence, \((1 - 5x + 6x^2)S(x) = s_0 + (s_1 - 5s_0)x = 1 - 7x\). The generating function is given by \(S(x) = \frac{1 - 7x}{1 - 5x + 6x^2}\).

Question

Do you see any relation between the denominator in the generating function above and the linear recurrence formula of the sequence?
Reciprocals of Polynomials

Definition 19

Let \( a(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \) be a polynomial. Its **reciprocal polynomial**, denoted by \( a^*(x) \), is defined by

\[
a^*(x) = a_n + x_{n-1} x + a_{n-2} x^2 + \cdots + a_0 x^n.
\]

Example 20

The reciprocal of \( a(x) = 1 + 3x + 2x^5 \) is \( a^*(x) = 2 + 3x^4 + x^5 \).
Theorem 21

Let \((s_i)_{i=0}^{\infty}\) be a sequence satisfying the following linear recurrence relation

\[ s_i = c_1 s_{i-1} + c_2 s_{i-2} + \cdots + c_\ell s_{i-\ell} \quad \text{for all } i \geq \ell, \]

where \(c_\ell \neq 0\). Then its generating function is given by \(S(x) = P(x)/Q(x)\), where

\[ Q(x) = 1 - c_1 x - c_2 x^2 - \cdots - c_\ell x^\ell, \]

which is the reciprocal of the characteristic polynomial of the sequence, and \(P(x)\) is some polynomial of degree less than \(\ell\).

Proof.

Define \(P(x) = S(x)Q(x)\). It is straightforward to determine \(P(x)\) and prove that its degree is at most \(\ell - 1\).
Theorem 22

Let

\[ Q(x) = 1 - c_1 x - c_2 x^2 - \cdots - c_\ell x^\ell, \]

where \( c_\ell \neq 0 \). Let \( P(x) \) be a polynomial of degree less than \( \ell \). If \( (s_i)_{i=0}^{\infty} \) is a sequence with generating function \( S(x) = P(x)/Q(x) \), then the sequence must satisfy the following linear recurrence relation

\[ s_i = c_1 s_{i-1} + c_2 s_{i-2} + \cdots + c_\ell s_{i-\ell} \text{ for all } i \geq \ell. \]

Proof.

The proof is straightforward and left as an exercise.
Most recurrence relations are not linear, and may be very hard to solve. However, some of them are solvable. In this case, there is no general approach to solving nonlinear recursions.

Example 23

Solve the recurrence relation \( s_i = s_{i-1} + i \) for all \( i \geq 1 \) with the initial condition \( s_0 = 0 \).

Example 24

Solve the recurrence relation \( s_i = s_{i-1} + i^2 \) for all \( i \geq 1 \) with the initial condition \( s_0 = 0 \).