Multiple random walks on complex networks: A harmonic law predicts search time

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Abstract

We investigate multiple random walks traversing independently and concurrently on complex networks and introduce the concept of mean first parallel passage time (MFPPT) to quantify their search efficiency. The mean first parallel passage time represents the expected time required to find a given target by one or some of the multiple walkers. We develop a general theory that allows us to calculate the MFPPT analytically. Interestingly, we find that the global MFPPT follows a harmonic law with respect to the global mean first passage times of the associated walkers. Remarkably, when the properties of multiple walkers are identical, the global MFPPT decays in a power law manner with an exponent of unity, irrespective of network structure. These findings are confirmed by numerical and theoretical results on various synthetic and real networks. The harmonic law reveals a universal principle governing multiple random walks on networks that uncovers the contribution and role of the combined walkers in target search. Our paradigm is also applicable to a broad range of random search processes.

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It has been recognized that random walk processes are a critical branch of network science [1]. Interest in these precesses originates from their broad relevance in various circumstances ranging from the spread of diseases [2] to animal foraging [3], from gene transcription [1] to transportation [4]. A fundamental quantity for characterizing random walks is first passage time (FPT), which quantifies the time a random walker needs to reach a given target. The intriguing properties of FPT have led to a growing number of investigations over the past decade including mean FPT on static networks [5] or in complex scale-invariant media [6] as well as on temporal networks [7] and the FPT distribution [8]. These studies not only enrich our knowledge of random walk processes on networks, but also facilitate the understanding of diverse dynamical processes such as synchronization [9], cascading failures [10], and zero-range processes [11].

However, previous studies of random walk processes have been confined to a single walker [5–8]. Multiple walkers moving concurrently, which are commonly encountered in real processes (examples include Knots in vibrated granular chains [12], record statistics of stock price [14], and a single prey survives around several predators [13]) have been less extensively considered. In fact, it is unambiguously agreed that multiple random walks can better imitate and model diverse dynamical processes. Taking disease and rumor spreading as an example, usually, several epidemics or rumors tend to break out in parallel. Nevertheless, the studies of multiple random walks remain in the early stages. Results are limited to specific examples [13, 14] or to bounds of the cover time [15]. A general framework with which to determine and characterize the search time of multiple random walks, however, has not been constructed yet.

In this paper, we develop a theoretical framework for characterizing multiple random walks traversing on networks and derive an analytical expression of mean first parallel passage time (MFPPT) — the expected time needed for reaching a given target by one or some of the multiple walkers. We find that the global MFPPT of multiple random walks is fully determined by the global mean first passage time (MFPT) of the associated walkers. Here, the global MFPPT represents the MFPPT averaged over all possible pairs of starting location and target node on a given network. Specifically, the global MFPPT follows a harmonic law with respect to the global MFPTs of the associated walkers. Remarkably, for identical walkers, the global MFPPT presents a power law behavior with an exponent of unity. Our theoretical predictions are confirmed by numerical results for multiple random walks on various synthetic and real networks. These findings, for the first time, uncover a general principle governing the search time of multiple random walks and are also applicable to various random search processes.

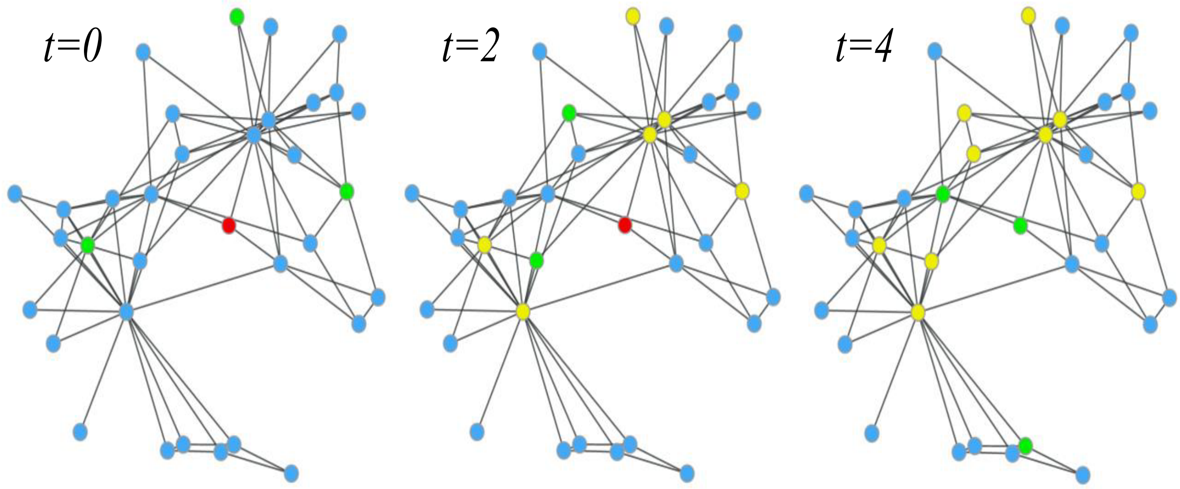

FIG. 1: (Color online) Schematic illustration of multiple random workers moving on the "Karate club" network [16]. The discrete time t denotes the number of steps of three walkers as they move on the network searching for a target node (red color). The green (yellow) colored nodes represent the positions of three walkers at time t (visited nodes in the past).

We start from m identical random walkers traveling independently and concurrently on a network consisting of N nodes. The connectivity of nodes is represented by an adjacency matrix A, whose entries $a_{ij} = 1$ (or 0) if there is (not) a link from nodes i to j. At each time step, each walker moves from current node i to node j with the transition probability p_{ij}, which is time-invariant value (i.e., memoryless). Take unbiased random walks for example, the transition probability is $p_{ij} = a_{ij}/\sum_j a_{ij}$. We denote by $T_{\Omega_m,j}$ the mean first parallel passage time defined as the expected time needed for any walker to reach node j starting from the nodes $\Omega_m = \{v_1, v_2, \cdots, v_m\}$ (see Fig.1). In this situation, if the first step of any walker is to node j, the expected number of steps required is 1; otherwise, all walkers are to some other nodes $\Omega'_m = \{v'_i | v'_i \neq j, i = 1, 2, \cdots, m\}$, the expected time becomes $T_{\Omega'_m,j} + 1$. Thus, we have

$$T_{\Omega_m,j} = \sum_{j\in\{v''_i\}_{i=1}^m} \prod_{i=1}^m p_{v_i,v''_i} + \sum_{j\notin\Omega'_m} \Psi(\Omega_m \to \Omega'_m)(T_{\Omega'_m,j} + 1), \tag{1}$$

where $\Psi(\Omega_m \to \Omega'_m) = \sum_{\{\tilde{v}_i\}_{i=1}^m \subset S(\Omega'_m)} \prod_{i=1}^m p_{v_i,\tilde{v}_i}$ represents the transition probability from the set Ω_m to the set Ω'_m for the walkers in one step and $S(\Omega'_m)$ is an ensemble of all possible combination deduced from the set Ω'_m. For example, $S(\Omega'_m) = \{\{1,2,3\},\{1,3,2\},\{2,1,3\},\{2,3,1\},\{3,1,2\},\{3,2,1\}\}$ for $\Omega'_m = \{1,2,3\}$. Since it is easy to verify that $\sum_{j\in\{v''_i\}_{i=1}^m}\prod_{i=1}^m p_{v_i,v''_i} + \sum_{j\notin\Omega'_m}\Psi(\Omega_m\to\Omega'_m) = 1$, Eq. (1) can be simplified as

$$T_{\Omega_m,j} = 1 + \sum_{j\notin\Omega'_m}\Psi(\Omega_m\to\Omega'_m)T_{\Omega'_m,j}. \tag{2}$$

When considering all possible initial condition (i.e., the position of starting nodes) for the m identical walkers, Eq. (2) can be rewritten in matrix form as

$$T_j^{(m)} = e + P^{(m)}T_j^{(m)}, \tag{3}$$

where e is a C_{N+m-2}^m-dimensional vector with all entries 1, $T_j^{(m)} = (T_{\{v_i=1\}_{i=1}^m,j}, T_{\{v_m=2,v_i=1\}_{i=1}^{m-1},j},\cdots,T_{\{v_i=N\}_{i=1}^m,j})^\top$ is a C_{N+m-2}^m-dimensional vector, and $P^{(m)}$ is a $C_{N+m-2}^m\times C_{N+m-2}^m$ matrix

$$P^{(m)} = \begin{pmatrix} 0 & \Psi(\{v_i=1\}_{i=1}^m\to\{v_m=2,v_i=1\}_{i=1}^{m-1}) & \cdots & \Psi(\{v_i=1\}_{i=1}^m\to\{v_i=N\}_{i=1}^m) \\ \Psi(\{v_m=2,v_i=1\}_{i=1}^{m-1}\to\{v_i=1\}_{i=1}^m) & 0 & \cdots & \Psi(\{v_m=2,v_i=1\}_{i=1}^{m-1}\to\{v_i=N\}_{i=1}^m) \\ \vdots & \vdots & \vdots & \vdots \\ \Psi(\{v_i=N\}_{i=1}^m\to\{v_i=1\}_{i=1}^m) & \Psi(\{v_i=N\}_{i=1}^m\to\{v_m=2,v_i=1\}_{i=1}^{m-1}) & \cdots & 0 \end{pmatrix}.$$

Note that the exact size C_{N+m-2}^m of the vector $T_j^{(m)}$ is obtained based on the famous identity (i.e., $\sum_{i=0}^{N-2}C_{i+m-1}^{m-1} = C_{N+m-2}^m$) in combinatorics. Since the matrix $(I-P^{(m)})$ has an inverse (see Appendixes A), we have

$$T_j^{(m)} = (I-P^{(m)})^{-1}e, \tag{4}$$

where I is an identity matrix. Equation (4) is important as it provides a universal formula for calculating MFPPT of multiple identical random walks moving on the network analytically. Specifically, when $m=1$, it is easy to verify that the MFPPT degenerates to the MFPT, to which the previous studies have been devoted [5–7]. Meanwhile, in the process of deriving the MFPPT, we only require the stochastic motion of multiple walkers satisfying the Markov property (i.e., memoryless) regardless of the exact form of the transition probability p_{ij}. Therefore, our analysis is applicable to a broad range of random search processes. Note that, in the process of multiple random search, several walkers may occupy the same node at some time t, as illustrated in Fig. 1. When neglecting node capacity on a network, the walker number m can be an arbitrary value. However, in practice, the node capacity is a limited value depending on the type of networks under consideration [17]. Here, for convenience, we assume the walker number $m \ll N$ for the rest of this paper.

We now confirm the analytical results by Monte Carlo simulations of multiple random walkers taking place in the "Karate club" network [16], where the walkers adopt the unbiased random walk strategy for target search. Fig. 2 (a) shows an excellent agreement between the analytical results and the numerical simulations. Here, we introduce the average parallel trapping time (APTT) to quantify the role of target location on multiple random walker search defined as follows:

$$\langle T_i^{(m)} \rangle = \frac{1}{C_{N+m-2}^{m}} \sum_{\Omega_m} T_{\Omega_m,i}. \tag{5}$$

Clearly, the APTT $\langle T_i^{(m)} \rangle$ represents the average of MFPPT $T_{\Omega_m,i}$ to the trap node i taken over all possibly starting points with the uniform distribution. Meanwhile, from Fig. 2 (a), we notice that the profiles of $\langle T_i^{(m)} \rangle$ versus i presents the same tendency for different walker numbers m. The result indicates that increasing the number of walkers just shifts vertically the curve of $\langle T_i^{(m)} \rangle$ versus i but does not affect the role of target location on trapping. This suggests that the trapping behavior of multiple identical random walks is comparable to that of a single walker as fully illustrated in Ref. [8].

In practice, we are more concerned about the effect of multiple walkers on search efficiency. To quantify the search efficiency at a global scale, we introduce the global MFPPT $\langle T^{(m)} \rangle$ by averaging MFPPT $T_{\Omega_m,i}$ over all possible pairs of starting and target nodes, that is

$$\langle T^{(m)} \rangle = \frac{1}{N C_{N+m-2}^{m}} \sum_{i} \sum_{\Omega_m} T_{\Omega_m,i}. \tag{6}$$

We investigate the behavior of the global MFPPT $\langle T^{(m)} \rangle$ on various networks including two typical synthetic models (the Barabási-Albert (BA) model [18] and the Erdös-Rényi (ER) model [19]) and five real networks (the "Karate club" network [16], the "Dolphin" network [20], the "Football" network [21], the "Chesapeake" network [22], and the "Polbooks" network [23]). Interestingly, we find the global MFPPT $\langle T^{(m)} \rangle$ follows a uniform power law with the walker number m such that $\langle T^{(m)} \rangle \sim m^{-1}$, as illustrated in Fig. 2(b). It reveals that the global MFPPT decays in a power law manner with an exponent of unity for multiple identical random walks, irrespective of network organization. Utilizing the annealed network approach [24] and the *Sherman − Morrison* formula [25], we present analytical arguments to demonstrate the uniform scaling behavior of $\langle T^{(m)}\rangle$ versus m (see Appendixes B)

$$\langle T^{(m)}\rangle \approx \frac{\langle T^{(1)}\rangle}{m}. \tag{7}$$

This is further supported by observing the behavior of multiply biased random walks on target search, where the transition probability is changed to $p_{ij} = \frac{a_{ij}k_j^\alpha}{\sum_j a_{ij}k_j^\alpha}$ with the tuning parameter α. Clearly, the prediction of Eq. (7) unambiguously captures the universal behavior of global MFPPT $\langle T^{(m)}\rangle(\alpha)$ versus m, as shown by the data collapse of analytical results calculated using Eq. (6) (see Fig. 2 (c) and (d)). Meanwhile, we notice that the global MFPPT $\langle T^{(m)}\rangle(\alpha)$ is minimized exactly when the global MFPT $\langle T^{(1)}\rangle(\alpha)$ for a single walker is minimized, as illustrated in Fig. 2 (c) and (d). These results shed new light on the role of the tuning parameter α in the optimization of search processes, and clearly point towards its robustness in multiple random search.

Moreover, the properties and behaviors of the adopted walkers are sometimes distinct, which is commonly encountered in various environments. For example, a prey species usually has several distinct predators in the wild. In this context, we consider m distinct walkers moving on the network independently and concurrently. Specially, for the l^{th} walker, at each time step, the transition probability from node i to node j is $p_{i,j}^{(l)}$. In this context, following the spirit of deriving the MFPPT for multiple identical random walks, we have (see Appendixes C)

$$T_j^{(m)} = (I - P^{(m)})^{-1} e, \tag{8}$$

where e is an $(N-1)^m$-dimensional vector with all entries 1, $T_j^{(m)} = (T_{\{v_i=1\}_{i=1}^m,j}, T_{\{v_m=2,v_i=1\}_{i=1}^{m-1},j}, \cdots, T_{\{v_i=N\}_{i=1}^m,j})^\top$ is an $(N-1)^m$-dimensional vector, and $P^{(m)}$ is an $(N-1)^m \times (N-1)^m$ matrix

$$P^{(m)} = \begin{pmatrix} \prod_{l=1}^m p_{1,1}^{(l)} & p_{1,2}^{(m)} \prod_{l=1}^{m-1} p_{1,1}^{(l)} & \cdots & \prod_{l=1}^m p_{1,N}^{(l)} \\ p_{2,1}^{(m)} \prod_{l=1}^{m-1} p_{1,1}^{(l)} & p_{2,2}^{(m)} \prod_{l=1}^{m-1} p_{1,1}^{(l)} & \cdots & p_{2,N}^{(m)} \prod_{l=1}^{m-1} p_{1,N}^{(l)} \\ \vdots & \vdots & \vdots & \vdots \\ \prod_{l=1}^m p_{N,1}^{(l)} & p_{N,2}^{(m)} \prod_{l=1}^{m-1} p_{N,1}^{(l)} & \cdots & \prod_{l=1}^m p_{N,N}^{(l)} \end{pmatrix}.$$

To test the validity of Eq. (8), we report both the numerical and theoretical results of the APTT $\langle T_i^{(m)}\rangle$ on the "Karate club" network. Fig. 3(a) shows that the simulation results match the analytical results of Eq. (8) accurately. Here, the distinct walkers are navigated by different search strategies. Specifically, when $m = 2$, we adopt two walkers executing an

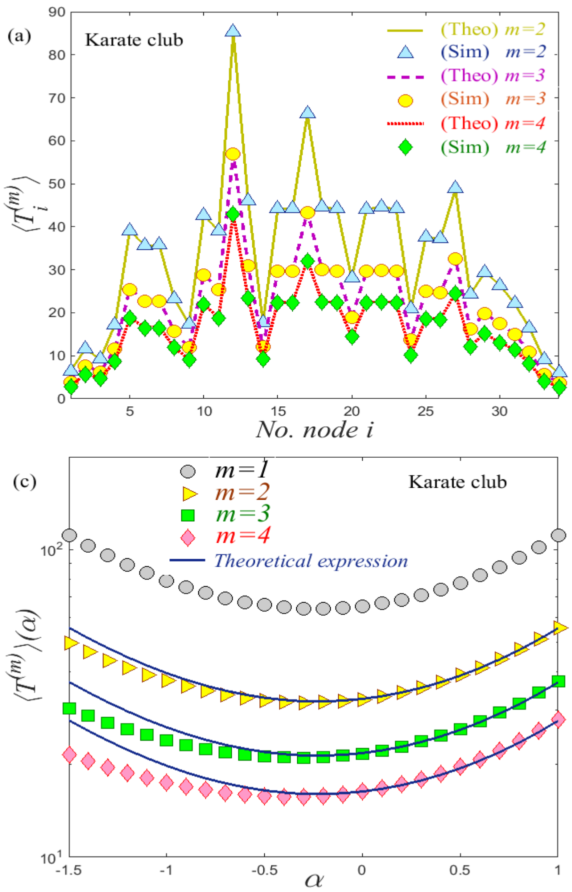

FIG. 2: (Color online) (a) The effect of target node location on multiple random walks search for the "Karate club" network [16]. All data collapse to the theoretical results given by Eq. (6). The simulation test is averaged over the ensemble of 400,000 independent runs. (b) The global MFPPT $\langle T^{(m)}\rangle$ as a function of the walker number m on various networks, where a universal scaling behavior emerges. The global MFPPT $\langle T^{(m)}\rangle(\alpha)$ for multiple biased random walks on (c) the "Karate club" network and (d) the "Chesapeake" network. The theoretical curves are given by Eq. (7).

unbiased random walk and a biased random walk with the tuning parameter $\alpha = 1.5$, while for $m = 3$, we add a new walker performing a biased random walk with $\alpha = 1$. Meanwhile, we notice that the profiles of $\langle T_i^{(m)}\rangle$ versus node index i present the same tendency, which implies that the effect of target position on trapping is independent of the number of distinct walkers.

Finally, we address a fascinating problem of how the search time is related to the properties of distinct walkers. Utilizing the annealed network approach [24] and the

$Sherman - Morrison$ formula [25], we obtain the harmonic law governing the global MFPPT $\langle T^{(m)} \rangle$ and the global MFPTs of distinct walkers (see Appendixes D)

$$\langle T^{(m)} \rangle \approx \frac{1}{\sum_l^m \frac{1}{\langle T^{(1)}(l) \rangle}}, \tag{9}$$

where $\langle T^{(1)}(l) \rangle$ is the global MFPT of the l^{th} walker. Specifically, when the walkers are identical, it is easy to verify that the harmonic law of Eq. (9) degenerates to the power law behavior of Eq. (7), as we expected. Remarkably, the harmonic law clearly reveals the contribution and role of the associated walkers on the search time of multiple random search.

To demonstrate the harmonic law of multiple random walks, we investigate the global MFPPT $\langle T^{(m)} \rangle$ on the previous selected networks. Here, we adopt biased random walks with different tuning parameters α to imitate the behaviors of distinct walkers. In fact, the tuning parameter α controls the preference for visiting high or low degree node at each time step, which in turn determines the behavior of biased random walks and influences their search efficiency. The results show that the analytic results are consistent with the theoretical prediction of Eq. (9), as illustrated in Fig. 3 (b). This indicates that the harmonic law of Eq. (9) unambiguously captures the search efficiency of multiple distinct walkers. Note that here, for the two distinct walkers, one adopts unbiased random walk strategy, while another performs biased random walk strategy with the tuning parameter α varying in the range $[-1.5, 1]$. Moreover, we investigate how the search efficiency of multiple random walks changes with respect to the tuning parameters of the walkers. Generally, regions with smaller $\langle T^{(2)} \rangle$ indicate an efficient way of multiple random search. Fig. 3 (c) and (d) shows contour maps of $\langle T^{(2)} \rangle$ in the (α_1, α_2) plane, where α_1 and α_2 are the tuning parameters of two distinct random walkers, respectively. Interestingly, we find that the (α_1, α_2) plane presents a "fingerprint" pattern, which further exhibits the harmonic characteristics of the global MFPPT $\langle T^{(2)} \rangle$ with respect to their associated walkers. These results to some extent demonstrate the uniform principle governing the search efficiency of multiple random walks.

In summary, we provide a generic paradigm for characterizing multiple random walks on networks and obtain the general expression of the mean first parallel passage time analytically. We introduce the global MFPPT to characterize the search efficiency of multiple random walks on networks at a global scale. Interestingly, we find that the global MFPPT of multiple random walks follows a harmonic law with respect to the MFPTs of the walkers.

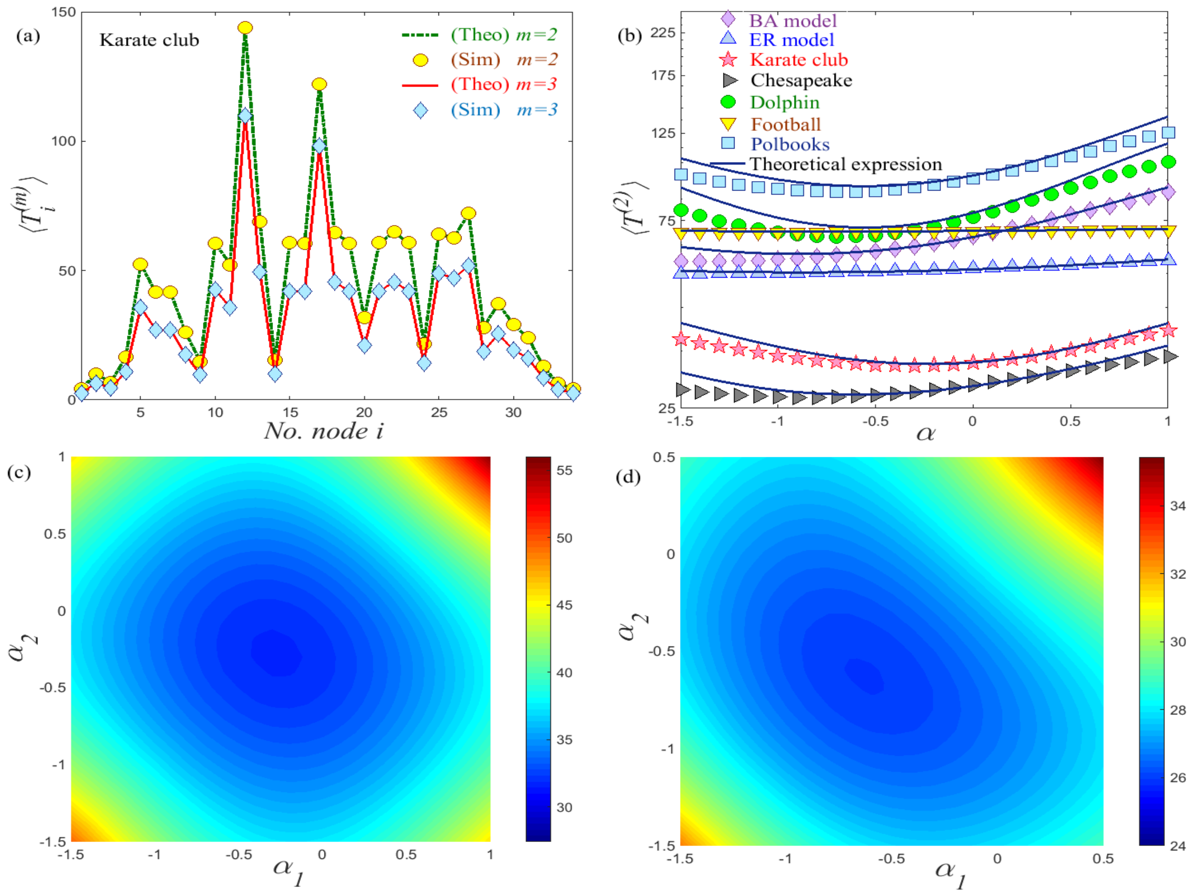

FIG. 3: (Color online) (a) The effect of target node selection on multiple distinct random searches on the "Karate club" network. All data collapse to the analytical results given by Eq. (8). The simulation test is averaged over the ensemble of 400,000 independent runs. (b) The global MFPPT $\langle T^{(2)} \rangle$ as a function of α for two random walkers. The theoretical curves are given by Eq. (9). The global MFPPT $\langle T^{(2)} \rangle$ in the (α_1, α_2) parameter plane of (c) the "Karate club" network and (d) the "Chesapeake" network, where α_1 and α_2 represent the tuning parameters of two associated biased random walkers.

Specifically, when the walkers are identical, the global MFPPT $\langle T^{(m)} \rangle$ decays in a power law manner with an exponent of unity. Our theoretical predictions are demonstrated by simulation and analytical results on various networks. Our findings, for the first time, uncover the underlying principle governing multiple random walks improving our understanding of diffusion processes. Our paradigm is applicable to a broad range of random search processes including intermittent search strategies [26], Lévy walks [27], and persistent random walks [28]. Note that here we assume that the multiple random walks are independent without sharing information in the process of target search. A more intriguing open problem is what happens when there is information sharing among multiple random walks in the search process.

I. APPENDIXES

A. The matrix $(I - P^{(m)})$ is nonsingular

For convenience, we denote by $L = I - P^{(m)}$, where I is an identity matrix and $P^{(m)}$ is a $C_{N+m-2}^m \times C_{N+m-2}^m$ matrix

$$P^{(m)} = \begin{pmatrix} 0 & \Psi(\{v_i=1\}_{i=1}^m \to \{v_m=2, v_i=1\}_{i=1}^{m-1}) & \cdots & \Psi(\{v_i=1\}_{i=1}^m \to \{v_i=N\}_{i=1}^m) \\ \Psi(\{v_m=2, v_i=1\}_{i=1}^{m-1} \to \{v_i=1\}_{i=1}^m) & 0 & \cdots & \Psi(\{v_m=2, v_i=1\}_{i=1}^{m-1} \to \{v_i=N\}_{i=1}^m) \\ \vdots & \vdots & \vdots & \vdots \\ \Psi(\{v_i=N\}_{i=1}^m \to \{v_i=1\}_{i=1}^m) & \Psi(\{v_i=N\}_{i=1}^m \to \{v_m=2, v_i=1\}_{i=1}^{m-1}) & \cdots & 0 \end{pmatrix}.$$

From the properties of matrices I and $P^{(m)}$, it is easy to check that L is a weakly diagonally dominant matrix, i.e., $|l_{ii}| \geq \sum_{k \neq i} |l_{ik}|$. Since node j is the target node and the network is connected, there exists several node sets $\{\Omega_m^{(i)} = \{v_1, v_2, \cdots, v_m\}\}_{i=1}^s$ such that $\prod_{i=1, j \in \{v_i''\}}^m p_{v_i, v_i''} > 0$. For notational convenience let us denote the node sets $\{\Omega_m^{(i)}\}_{i=1}^s$ in the matrix $P^{(m)}$ as $\{z_i\}_{i=1}^s$. Thus, for the row z_i, it satisfies $|l_{z_i z_i}| > \sum_{k \neq z_i} |l_{z_i k}|$, called a strictly diagonally dominant row. Suppose there exists a nonzero vector $X = (x_1, x_2, \cdots, x_{N+m-2})$ in the null space of L. Let i_1 be such that $|x_{i_1}| \geq |x_k|$ for all k. Then, we have

$$l_{i_1 i_1} x_{i_1} = \sum_{k \neq i_1} l_{i_1 k} x_k. \tag{10}$$

Applying the triangle inequality yields

$$l_{i_1 i_1} \leq \sum_{k \neq i_1} |l_{i_1 k}| \, |x_k| \, / \, |x_{i_1}| \leq \sum_{k \neq i_1} |l_{i_1 k}|. \tag{11}$$

Since L is a weakly diagonally dominant matrix, the above inequality means that $|x_k| = |x_{i_1}|$ whenever $l_{i_1 k} \neq 0$. As the network is a connected graph, we can pick an index sequence $\Omega_m^{(i_1)} \to \Omega_m^{(i_2)} \to \cdots \to \Omega_m^{(i_s)}$ ending at some row z_i in the matrix $P^{(m)}$ with the property $\Psi(\Omega^{(i_k)} \to \Omega^{(i_{k+1})}) > 0$. Therefore $|x_{i_2}| = |x_{i_1}|$. Repeating this argument with i_2, i_3, etc., we find that z_i is not strictly diagonally dominant row, a contradiction. Thus, we proof that the matrix $(I - P^{(m)})$ is nonsingular.

B. The universal behavior between MFPPT and the number of identical walkers

Now, we address how the effect of increasing the number of multiple walkers on the efficiency of target search. For an uncorrelated network, we can reinterpret its adjacency matrix A as a weighted fully connected graph $\widetilde{A}$ using the annealed network approach [24]. In particular, the entry $\widetilde{a}_{ij}$ defines the connection probability between nodes i and j, namely

$$\widetilde{a}_{ij} = \frac{k_i k_j}{N\langle k\rangle}, \tag{12}$$

where $\langle k\rangle$ is the average degree of the whole network. In this situation, the adjacency matrix $\widetilde{A}$ can be represented as

$$\widetilde{A} = \frac{\boldsymbol{K}\boldsymbol{K}^T}{N\langle k\rangle}, \tag{13}$$

where $\boldsymbol{K} = (k_1, k_2, \cdots, k_N)^T$ is the degree sequence. Here, we assume that the degree sequence of the network follows a Poisson distribution. In the process of biased random walks on the weighted network $\widetilde{A}$, the transition probability P becomes

$$P = \frac{1}{\sum_{i=1}^{N} k_i^{1+\alpha}} \begin{pmatrix} k_1^{1+\alpha} & k_2^{1+\alpha} & \cdots & k_N^{1+\alpha} \\ k_1^{1+\alpha} & k_2^{1+\alpha} & \cdots & k_N^{1+\alpha} \\ \vdots & \vdots & \vdots & \vdots \\ k_1^{1+\alpha} & k_2^{1+\alpha} & \cdots & k_N^{1+\alpha} \end{pmatrix}. \tag{14}$$

Substituting the above equation into the matrix $P^{(m)}$, we have

$$P^{(m)} = \frac{1}{\left(\sum_{i=1}^{N} k_i^{1+\alpha}\right)^m} \begin{pmatrix} k_1^{m(1+\alpha)} & m k_1^{1+\alpha} k_2^{(m-1)(1+\alpha)} & \cdots & m! k_{j-1}^{1+\alpha} \prod_{l=N+2-m}^{N} k_l^{1+\alpha} & k_{j+1}^{m(1+\alpha)} & \cdots & k_N^{m(1+\alpha)} \\ k_1^{m(1+\alpha)} & m k_1^{1+\alpha} k_2^{(m-1)(1+\alpha)} & \cdots & m! k_{j-1}^{1+\alpha} \prod_{l=N+2-m}^{N} k_l^{1+\alpha} & k_{j+1}^{m(1+\alpha)} & \cdots & k_N^{m(1+\alpha)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ k_1^{m(1+\alpha)} & m k_1^{1+\alpha} k_2^{(m-1)(1+\alpha)} & \cdots & m! k_{j-1}^{1+\alpha} \prod_{l=N+2-m}^{N} k_l^{1+\alpha} & k_{j+1}^{m(1+\alpha)} & \cdots & k_N^{m(1+\alpha)} \end{pmatrix}.$$

Utilizing the *Sherman − Morrison* formula [25], the inverse of the matrix $(I - P^{(m)})$ become

$$(I - P^{(m)})^{-1} = I + \frac{1}{\left(\sum_{i=1}^{N} k_i^{1+\alpha}\right)^m - \left(\sum_{i=1}^{N} k_i^{1+\alpha} - k_j^{1+\alpha}\right)^m} P^{(m)}. \tag{15}$$

Inserting the above equation into Eq. (3), we have

$$T_j^{(m)} = \frac{\left(\sum_{i=1}^{N} k_i^{1+\alpha}\right)^m}{\left(\sum_{i=1}^{N} k_i^{1+\alpha}\right)^m - \left(\sum_{i=1}^{N} k_i^{1+\alpha} - k_j^{1+\alpha}\right)^m} e, \tag{16}$$

where e is a C_{N+m-2}^{m}-dimensional vector with all entries 1. Substituting Eq. (16) into Eq. (6) and using the second-order Taylor expansion of the term $\left(\sum_{i=1}^{N} k_i^{1+\alpha} - k_j^{1+\alpha}\right)^m$, we have

$$\langle T^{(m)} \rangle \approx \frac{\sum_{i=1}^{N} k_i^{1+\alpha}}{Nm} \sum_{i=1}^{N} \frac{1}{k_i^{1+\alpha}}. \tag{17}$$

Since it is easy to verify that $\langle T^{(1)} \rangle = \frac{\sum_{i=1}^{N} k_i^{1+\alpha}}{N} \sum_{i=1}^{N} \frac{1}{k_i^{1+\alpha}}$ derived from the adjacency matrix $\widetilde{A}$, Eq. (17) can be solved to obtain

$$\langle T^{(m)} \rangle \approx \frac{\langle T^{(1)} \rangle}{m}. \tag{18}$$

Equation (18) reveals a linear relationship between the global MFPPT and the number of multiple walkers, meaning that increasing multiple random walkers can reduce the search time in a linear manner at a global scale. Note that, for rigorousness, the degree distribution of the network is assumed to follow a Poisson distribution, which ensures the approximation of using the second-order Taylor expansion is meaningful. But the same conclusion (i.e., Eq. (18)) holds in general even on scale-free networks (refer to the numerical findings in Fig. 2(b)).

C. The analytical expression of the mean first parallel passage time for multiple distinct random walks

Furthermore, we consider m distinct walkers traversing on the network independently and in parallel. For the l^{th} walker, at each time step, the transition probability from current node i to node j is $p_{i,j}^{(l)}$. In this context, we concern about how long do these walkers take to reach a given target? Without loss of generality, we assume that the target is placed at node j and the walkers are initially located at the nodes $\Omega_m = \{v_1, v_2, \cdots, v_m\}$. In this situation, if the first step of any walker is to node j, the expected number of steps required is 1; if all of them are to some other nodes $\Omega'_m = \{v'_i | v'_i \neq j, i = 1, 2 \cdots, m\}$, the expected time becomes $T_{\Omega'_m,j} + 1$. Thus, we have

$$T_{\Omega_m,j} = \sum_{j \in \{v''_i\}_{i=1}^{m}} \prod_{i=1}^{m} p_{v_i,v''_i}^{(i)} + \sum_{j \notin \Omega'_m} \prod_{i=1}^{m} p_{v_i,v'_i}^{(i)} (T_{\Omega'_m,j} + 1). \tag{19}$$

Since $\sum_{j \in \{v''_i\}_{i=1}^{m}} \prod_{i=1}^{m} p_{v_i,v''_i}^{(i)} + \sum_{j \notin \Omega'_m} \prod_{i=1}^{m} p_{v_i,v'_i}^{(i)} = 1$, equation (19) can be rewritten in matrix form as

$$T_j^{(m)} = e + P^{(m)} T_j^{(m)}, \tag{20}$$

where $T_j^{(m)} = (T_{\{v_i=1\}_{i=1}^m,j}, T_{\{v_m=2,v_i=1\}_{i=1}^{m-1},j}, \cdots, T_{\{v_i=N\}_{i=1}^m,j})^\top$ is an $(N-1)^m$-dimensional vector and $P^{(m)}$ is

$$P^{(m)} = \begin{pmatrix} \prod_{i=1}^m p_{1,1}^{(i)} & p_{1,2}^{(m)} \prod_{i=1}^{m-1} p_{1,1}^{(i)} & \cdots & \prod_{i=1}^m p_{1,N}^{(i)} \\ p_{2,1}^{(m)} \prod_{i=1}^{m-1} p_{1,1}^{(i)} & p_{2,2}^{(m)} \prod_{i=1}^{m-1} p_{1,1}^{(i)} & \cdots & p_{2,N}^{(m)} \prod_{i=1}^{m-1} p_{1,N}^{(i)} \\ \vdots & \vdots & \vdots & \vdots \\ \prod_{i=1}^m p_{N,1}^{(i)} & p_{N,2}^{(m)} \prod_{i=1}^{m-1} p_{N,1}^{(i)} & \cdots & \prod_{i=1}^m p_{N,N}^{(i)} \end{pmatrix}_{(N-1)^m \times (N-1)^m}.$$

Following a straightforward generalization of the proof carried out in Appendix A, we can prove that the matrix $(I - P^{(m)})$ is reversible, where I is an identity matrix. Then, we have

$$T_j^{(m)} = (I - P^{(m)})^{-1} e. \tag{21}$$

The equation (21) provides a universal formula for calculating mean first parallel passage time of multiple distinct random walks finding a given target analytically.

D. The harmonic law for multiple distinct random walks

Finally, we examine the contribution of each walker to the efficiency of multiple distinct random walkers for target search. Similarly, using the annealed network approach for reinterpreting the adjacency matrix [24] and substituting the transition probability P (i.e., Eq. (14)) into the matrix $P^{(m)}$, we obtain

$$P^{(m)} = \begin{pmatrix} \frac{\prod_{l=1}^m k_1^{1+\alpha_l}}{\prod_{l=1}^m \sum_{i=1}^N k_i^{1+\alpha_l}} & \frac{k_2^{1+\alpha_m} \prod_{l=1}^{m-1} k_1^{1+\alpha_l}}{\prod_{l=1}^m \sum_{i=1}^N k_i^{1+\alpha_l}} & \cdots & \frac{k_{j-1}^{1+\alpha_1} \prod_{l=2}^m k_N^{1+\alpha_l}}{\prod_{l=1}^m \sum_{i=1}^N k_i^{1+\alpha_l}} & \frac{k_{j+1}^{1+\alpha_1} \prod_{l=2}^m k_1^{1+\alpha_l}}{\prod_{l=1}^m \sum_{i=1}^N k_i^{1+\alpha_l}} & \cdots & \frac{\prod_{l=1}^m k_N^{1+\alpha_l}}{\prod_{l=1}^m \sum_{i=1}^N k_i^{1+\alpha_l}} \\ \frac{\prod_{l=1}^m k_1^{1+\alpha_l}}{\prod_{l=1}^m \sum_{i=1}^N k_i^{1+\alpha_l}} & \frac{k_2^{1+\alpha_m} \prod_{l=1}^{m-1} k_1^{1+\alpha_l}}{\prod_{l=1}^m \sum_{i=1}^N k_i^{1+\alpha_l}} & \cdots & \frac{k_{j-1}^{1+\alpha_1} \prod_{l=2}^m k_N^{1+\alpha_l}}{\prod_{l=1}^m \sum_{i=1}^N k_i^{1+\alpha_l}} & \frac{k_{j+1}^{1+\alpha_1} \prod_{l=2}^m k_1^{1+\alpha_l}}{\prod_{l=1}^m \sum_{i=1}^N k_i^{1+\alpha_l}} & \cdots & \frac{\prod_{l=1}^m k_N^{1+\alpha_l}}{\prod_{l=1}^m \sum_{i=1}^N k_i^{1+\alpha_l}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\prod_{l=1}^m k_1^{1+\alpha_l}}{\prod_{l=1}^m \sum_{i=1}^N k_i^{1+\alpha_l}} & \frac{k_2^{1+\alpha_m} \prod_{l=1}^{m-1} k_1^{1+\alpha_l}}{\prod_{l=1}^m \sum_{i=1}^N k_i^{1+\alpha_l}} & \cdots & \frac{k_{j-1}^{1+\alpha_1} \prod_{l=2}^m k_N^{1+\alpha_l}}{\prod_{l=1}^m \sum_{i=1}^N k_i^{1+\alpha_l}} & \frac{k_{j+1}^{1+\alpha_1} \prod_{l=2}^m k_1^{1+\alpha_l}}{\prod_{l=1}^m \sum_{i=1}^N k_i^{1+\alpha_l}} & \cdots & \frac{\prod_{l=1}^m k_N^{1+\alpha_l}}{\prod_{l=1}^m \sum_{i=1}^N k_i^{1+\alpha_l}} \end{pmatrix},$$

where α_l is the tuning exponent of the l^{th} walker. Utilizing the $Sherman - Morrison$ formula [25], the inverse of the matrix $(I - P^{(m)})$ becomes

$$(I - P^{(m)})^{-1} = I + \frac{1}{\prod_{l=1}^m \sum_{i=1}^N k_i^{1+\alpha_l} - \prod_{l=1}^m \left(\sum_{i=1}^N k_i^{1+\alpha_l} - k_j^{1+\alpha_l} \right)} P^{(m)}. \tag{22}$$

Substituting the above equation into Eq. (21), we have

$$T_j^{(m)} = \frac{\prod_{l=1}^m \sum_{i=1}^N k_i^{1+\alpha_l}}{\prod_{l=1}^m \sum_{i=1}^N k_i^{1+\alpha_l} - \prod_{l=1}^m \left(\sum_{i=1}^N k_i^{1+\alpha_l} - k_j^{1+\alpha_l} \right)} e. \tag{23}$$

Here, we assume that the degree sequence of the network follows a Poisson distribution. Inserting Eq. (23) into Eq. (6) and using the second-order Taylor expansion of the term $\left(\prod_{l=1}^{m}\sum_{i=1}^{N}k_i^{1+\alpha_l}-\prod_{l=1}^{m}\left(\sum_{i=1}^{N}k_i^{1+\alpha_l}-k_j^{1+\alpha_l}\right)\right)$, we obtain

$$\langle T^{(m)}\rangle \approx \frac{1}{N}\sum_{j=1}^{N}\left(\frac{1}{\sum_{l=1}^{m}\frac{k_j^{1+\alpha_l}}{\sum_{i=1}^{N}k_i^{1+\alpha_l}}}\right). \tag{24}$$

Meanwhile, it is easy to verify that the global MFPT for the l^{th} walker is $\langle T^{(1)}(l)\rangle = \frac{\sum_{i=1}^{N}k_i^{1+\alpha_l}}{N}\sum_{i=1}^{N}\frac{1}{k_i^{1+\alpha_l}}$. Thus, equation (24) can be solved to obtain

$$\langle T^{(m)}\rangle \approx \frac{1}{\sum_{l=1}^{m}\frac{1}{\langle T^{(1)}(l)\rangle}}. \tag{25}$$

Equation (25) reveals that the global MFPPT of multiple distinct random walks follows the harmonic law with respect to the global MFPTs of the associated walkers. In particular, when the walkers are identical, it is easy to prove that Eq. (25) reduces to Eq.(18), as expected.

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