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Importance sampling for integrated market and credit portfolio models

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Abstract

A sophisticated approach for computing the total economic capital needed for various stochastically dependent risk types is the bottom-up approach. In this approach, usually, market and credit risks of financial instruments are modeled simultaneously. As integrating market risk factors into standard credit portfolio models increases the computational burden of calculating risk measures, it is analyzed to which extent importance sampling techniques previously developed either for pure market portfolio models or for pure credit portfolio models can be successfully applied to integrated market and credit portfolio models. Specific problems which arise in this context are discussed. The effectiveness of these techniques is tested by numerical experiments for linear and non-linear portfolios.

Keywords: Bottom-up approach; Credit risk; Importance sampling; Interest rate risk; Risk management; Value-at-risk

1. Introduction

Due to their business activities, banks are exposed to many different risk types. Among these risk types are credit risk, market risk, operational risk, and business risk. The task of the risk management division is to measure all these risks and to determine the necessary amount of economic capital which is needed as a buffer to absorb unexpected losses associated with each of these risks. Most frequently, economic capital is understood as a Value-at-Risk (VaR) number. Thus, it is the amount of capital needed to absorb unexpected losses within a given time period up to a specified probability.

Predominantly, the necessary amount of economic capital is determined for each risk type separately. That is why the problem arises how to combine these various amounts of capital to a single number. Within the so-called building–block approach stipulated by the regulatory authorities, the amount of regulatory capital, which the banks have to hold for the different risk types, are just added. This is a quite conservative approach because it ignores diversification effects between the risk types. As a consequence, in general, the true amount of economic or regulatory capital that is needed is overestimated.

However, the alternative, namely to consider diversification effects to some extent, requires to model the stochastic dependence between the various risk types. In practice, some kind of heuristics, based on strong assumptions, are often used to merge the economic capital figures for the various risk types into one overall economic capital figure.[1] A theoretical more sound approach is to link the separately determined marginal distributions of losses resulting from different risk types by Copula functions (see, e.g. Ward and Lee (2002), Dimakos and Aas (2004), Rosenberg and Schuermann (2006)). However, the difficulty is to choose the correct Copula function, especially given the limited access to adequate data.

Another more sophisticated approach is to build up models for various risk types by integrating a specific risk type into existing models for the measurement of another risk type. This approach is pursued in this paper, which deals, more

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[1] For an overview on risk aggregation methods used in practice, see Joint Forum (2003), Bank of Japan (2005), and Rosenberg and Schuermann (2006).

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specifically, with the integration of market risk into credit portfolio models. Integrated market and credit portfolio models allow to determine simultaneously, in one common framework, the necessary amount of economic capital needed for the market risk and for the credit risk of banking book instruments, whereby possible stochastic dependencies between these two risk components can be taken into account directly. This latter approach is called a bottom-up approach, whereas the Copula-based approach represents a top-down technique.

For measuring the credit risk inherent in a banking book, a range of models has been developed. Well-known examples are CreditMetrics by J.P. Morgan Chase, CreditPortfolioView by McKinsey, Portfolio Manager by KMV, or CreditRisk$^+$ by Credit Suisse First Boston. A typical shortcoming of most credit portfolio models is that relevant market risk factors, such as risk-free interest rates or credit spreads, are not modeled as stochastic variables and hence are ignored during the revaluation of the credit sensitive instruments at the risk horizon. An exception is the approach Algo Credit developed by the risk management firm Algorithmics (see Iscoe et al. (1999)). Even the Basel II proposals do not regulate the interest rate risk of the banking book in a quantitative way, but only qualitatively under pillar II (see Basel Committee on Banking Supervision (2005)). In a typical credit portfolio model, fixed-income instruments, such as bonds or loans, are revalued at the risk horizon using the current forward rates and (rating-specific) forward credit spreads for discounting future cash flows. Even for derivatives with counterparty risk, only single values, so-called loan equivalents, are employed per possible rating grade of the counterparty at the risk horizon. Thus, the stochastic nature of the instrument's value in the future that results from changes in factors other than credit quality is ignored. This may cause an underestimation of the riskiness of the credit portfolio (see, e.g. Barnhill and Maxwell (2002), Kiesel et al. (2003), Grundke (2005)). An additional consequence is that correlations between changes of the debtors' credit quality and changes of market risk factors and hence the exposure at default cannot be integrated into the credit portfolio model. This is especially a problem for market-driven instruments, such as interest rate derivatives, because the exposure at default mainly depends on the stochastic evolution of the underlying market risk factors. Finally, ignoring relevant market risk factors in credit portfolio models, correlations between the exposures at default of different instruments, which depend on the same or correlated market risk factors, cannot be modeled, either.

However, adding market risk factors as additional ingredients of a credit portfolio model, the computational burden of calculating risk measures increases because the revaluation of the instruments at the risk horizon becomes more complex. Most standard credit portfolio models rely on Monte Carlo simulations for calculating the probability distribution of the future credit portfolio value.[2] This is already computer-time-consuming for standard credit portfolio models, especially for inhomogeneous portfolios with many obligors and when percentiles corresponding to high confidence levels have to be estimated. Thus, the need for efficient methods for calculating credit risk measures becomes even more pressing for integrated market and credit portfolio models.

For standard credit portfolio models, various efficiency enhancing computational approaches have been developed meanwhile. Among these are, for example, approaches based on Monte Carlo simulations combined with variance reduction techniques, mainly importance sampling (IS) (for a literature review see the next Section 2), Fourier-based approaches (see Duffie and Pan (2001), Merino and Nyfeler (2002), Reiß (2003)), computational approaches based on saddlepoint approximations (see, e.g. Arvanitis et al. (1998), Martin et al. (2001a,b), Gordy (2002), Barco (2004)), or methods which rely on the assumption that the portfolio is sufficiently large or sufficiently granular so that by the virtue of the (strong) Law of Large Numbers (or the Central Limit Theorem) approximations of the credit portfolio loss variable are possible (see, e.g. Finger (1999), Vasicek (1991, 2002), Gordy (2003)).

For integrated market and credit portfolio models, it might suggest itself to simply adjust and to apply these techniques also to this extended class of portfolio models. However, it has already been reported in the literature that this simple strategy does not always work. For example, Grundke (2007) finds that the Fourier-based approach when applied to an integrated market and credit portfolio model for estimating risk measures does not perform better than a crude Monte Carlo simulation. In this paper, we analyze the performance of IS as a special variance reduction technique. Monte Carlo simulations combined with IS are reported to be very flexible in the computation of overall risk measures as well as individual risk contributions. We transfer an IS technique previously developed for pure market portfolio models as well as an IS technique previously developed for pure credit portfolio models to the case of integrated market and credit portfolio models. We find that, when applying these techniques to the extended class of integrated market and credit portfolio models, specific problems arise which reduce their benefit. These problems are discussed and the effectiveness of these techniques is tested by numerical experiments for linear and non-linear portfolios.

The paper is structured as follows: in Section 2, an overview on related literature is given. In Section 3, a general framework for an integrated market and credit portfolio model is presented. Besides, a concrete specification of this general model is described, which afterwards is used for the numerical experiments. In Section 4, two IS techniques are applied to the general integrated market and credit portfolio model. The first one is a two-step-technique originally developed

[2] A prominent exception is the model CreditRisk$^+$ where due to specific assumptions the loss distribution can be computed by recursion.

for a pure credit portfolio model. The second one is based on a delta–gamma approximation of the portfolio loss and was originally developed for a pure market risk portfolio model. This technique is combined with the credit risk-related two-step-IS technique. The effectiveness of the presented IS techniques is tested by means of numerical experiments in Section 5. Finally, in Section 6, the main results are summarized and possible extensions of this study are outlined.

2. Review of the literature

Most approaches based on Monte Carlo simulations combined with variance reduction techniques employ IS to accelerate the computation of credit risk measures. For a CreditMetrics-style, pure default mode credit portfolio model, Glasserman and Li (2005) employ IS for the probability distribution of the systematic risk factors as well as for the conditional default probabilities to estimate excess probabilities more efficiently than with naïve Monte Carlo simulations. In contrast to almost all other approaches, Glasserman and Li (2005) employ a two-step-IS procedure. The same technique is applied to a CreditRisk$^+$-style credit portfolio model by Glasserman and Li (2003). Glasserman (2006) shows how this approach can be employed for estimating obligor-specific credit risk contributions. Dunkel and Weber (2005) employ the same approach as Glasserman and Li (2005) for estimating utility-based shortfall risk measures in the framework of the credit portfolio model CreditRisk$^+$ and the default mode version of CreditMetrics.

Merino and Nyfeler (2004) only use IS for the conditional default indicators, but leave the probability distribution of the systematic risk factors unchanged. They employ the 'exponential twisting' technique (see Section 4.1 and Glasserman (2004, p. 260)), which is also applied by Glasserman and Li (2005). They demonstrate the use of this technique for estimating individual risk contributions (based on the expected shortfall) within a classical 'conditional independence'-framework with default mode. In a numerical example, they choose a specification of the credit portfolio model in the spirit of CreditRisk$^+$.

Another branch of papers solely deals with IS for the systematic risk factors. These papers mainly differ in the way how an effective IS distribution for the systematic risk factors is determined. Within the framework of the default mode CreditMetrics model, Kalkbrener et al. (2004) try to find optimal means of the systematic credit risk factors under the IS distribution by approximating the original inhomogeneous, finite portfolio by a homogeneous, infinitely granular portfolio. Then, they calculate the mean of the systematic credit risk factor that minimizes the variance of the estimator of the desired risk measure in a one-factor model of the homogeneous, infinitely granular portfolio. Finally, they 'lift' this one-dimensional optimal mean to a M-dimensional mean vector. They use this technique for estimating individual risk contributions based on the expected shortfall. An approximation of the original portfolio by an infinitely granular, homogeneous portfolio is also employed by Tchistiakov et al. (2004) to reduce the variance of the risk measure estimator. However, they employ this approximation as a control variate.[3] In a default mode version of CreditMetrics, Egloff et al. (2005) compute the means of the systematic risk factors under the IS distribution by an adaptive stochastic approximation procedure. Instead of modifying the means of the systematic risk factors, Morokoff (2004) scales up the variances and covariances of the asset returns. Under the IS probability measure, the asset returns are sampled according to a normal distribution with mean zero and a modified covariance matrix, which results from scaling up the largest eigenvalue from the eigenvalue–eigenvector decomposition of the original covariance matrix. Morokoff (2004) works in a default mode version of CreditMetrics.

Bassamboo et al. (2006) also employ a 'conditional independence'-framework with default mode, but – in contrast to other papers – they allow for an extremal dependence between the obligors, which is for example induced by modeling the asset returns with a multivariate t-distribution. They propose two ways for deriving an IS distribution for the random variable that causes common shocks for all obligors and hence dependence amongst them. As Bassamboo et al. (2006) also suggest to use 'exponential twisting' for the conditional default probabilities, their approach is another two-step-IS technique.

For the pricing of collateralized debt obligations, Joshi (2004) demonstrates how to apply IS by shifting the mean of the systematic risk factor. Joshi and Kainth (2004) also use this technique for computing the sensitivity of the price of a nth to default swap to changes in the underlying hazard rate of a particular obligor. These deltas are needed for hedging nth to default swaps.

In this paper, first, the two-step-IS technique presented by Glasserman and Li (GL) (2005) for a pure default mode model is applied to a general integrated market and credit portfolio model. It is pointed out which differences and difficulties result from the integration of market risk.[4] As already mentioned, GL (2005) employ IS for the probability distribution of the systematic risk factors as well as for the conditional default probabilities. Due to this two-step-nature of the IS technique, the approach suggested by GL (2005) is expected to be especially effective. Second, it is discussed how an IS

[3] Arvanitis et al. (1998) and Arvanitis and Gregory (2001, p. 83) also work with a control variate, which is, however, based on saddlepoint approximations.

[4] A less detailed analysis can be found in Grundke (2007).

approach originally developed by Glasserman et al. (2000) for pure market risk portfolio models can be combined with the two-step-IS approach. Glasserman et al. (2000) use a delta–gamma approximation of the loss variable of a portfolio of default risk-free instruments for selecting an effective IS distribution for the normally distributed vector of market risk factors. As the topic of this paper are integrated *market and credit* portfolio models, the idea to combine methods originally developed for pure market risk portfolio models with those originally developed for pure (default mode) credit portfolio models might suggest itself. However, up to now, this has not been tried.

Summarizing, the main questions answered in this paper are:

(1) Are IS techniques originally developed for pure default mode credit portfolio models also applicable to integrated market and credit portfolio models?
(2) How effective are they for these extended models?
(3) Is it possible to increase the effectiveness by combining IS techniques originally developed for pure default mode credit portfolio models with those originally developed for pure market risk portfolio models?

3. The integrated market and credit portfolio model

3.1. General approach

It is assumed that the credit portfolio consists of N market and credit risk sensitive instruments issued by N different corporates. The risk horizon of the credit portfolio model is denoted by H. P denotes the real world probability measure. The number of possible credit qualities at the risk horizon is K: one denotes the best rating and K is the default state.

The central part of most credit portfolio models is the definition of the obligors' conditional default and transition probabilities. Denoting by $\eta_H^n \in \{1,\ldots,K\}$ the credit quality of obligor n at the risk horizon H and by η_0^n the respective rating at $t=0$, the conditional default (transition) probabilities are formally defined as[5]:

$$P\left(\eta_H^n = k | \eta_0^n = i, Z_1 = z_1, \ldots, Z_C = z_C\right) := f_{n,i,k}(z_1,\ldots,z_C) \tag{1}$$

with

$$f_{n,i,k} : \mathbb{R}^C \to [0,1] \ (k \in \{1,\ldots,K\}, i \in \{1,\ldots,K-1\}, n \in \{1,\ldots,N\}).$$

The set of variables $Z=(Z_1,\ldots,Z_C) \sim G^C$ are systematic credit risk factors (e.g. changes in equity indices or macro-economic variables until the risk horizon). They influence the credit quality changes of all obligors. This vector is assumed to evolve according to a multivariate distribution G^C. Given the realization $(Z_1=z_1,\ldots,Z_C=z_C)$ of the systematic credit risk factors and hence of the conditional default (transition) probabilities, credit quality changes of all obligors are assumed to be stochastically independent. Thus, this is the classical 'conditional independence'-framework for describing joint credit quality changes in a credit portfolio.

The price of the instrument i_n (e.g. a defaultable (zero) coupon bond or an option with counterparty risk) at the risk horizon H, whose issuer n has not already defaulted before H and exhibits the rating $\eta_H^n \in \{1,\ldots,K-1\}$, is denoted by

$$p_n(\eta_H^n; X; P_n). \tag{2}$$

The stochastic vector $X=(X_1,\ldots,X_M) \sim G^M$ represents the value of relevant market risk factors at the risk horizon. This vector is assumed to evolve according to a multivariate distribution G^M. P_n denotes a vector of additional parameters relevant for the pricing of the respective instrument i_n at the risk horizon. Note that the set of systematic credit risk factors $Z_1,\ldots,Z_C$ and the set of market risk factors $X_1,\ldots,X_M$ can overlap, e.g., if a risk-free interest rate is also a relevant credit risk driver. The joint distribution of the stochastic vector $(Z_1,\ldots,Z_C, X_1,\ldots,X_M)$ is denoted by G. Later for the numerical examples, it will be assumed that G is a multivariate normal distribution.

If the issuer n of the instrument i_n has already defaulted $(\eta_H^n = K)$ until the risk horizon H, its value, in the case this value is positive, is set equal to a fraction δ_n of the value that the instrument would have at the risk horizon when its issuer would be free of default risk. If the market value of this instrument is negative, nothing is changed because the bank whose credit portfolio is considered is a debtor of the defaulted issuer. The recovery rate δ_n can vary with the seniority of a claim and the value of individual collaterals. Usually, it is assumed that the recovery rate is beta-distributed and independent from all other stochastic variables of the respective model, such as the systematic credit risk drivers or the market risk factors, but it could also be a function of these risk factors (see, e.g. Frye (2000), Pykhtin (2003)). Finally, the value $\Pi(H)$ of the entire portfolio at the risk horizon H is just the sum of the individual values:

[5] For this general modelling approach, see also Grundke (2007).

$$\Pi(H) = \sum_{n=1}^{N} p_n(\eta_H^n; X; P_n). \tag{3}$$

The credit portfolio loss variable is defined as the sum of the differences between the instrument's value at the risk horizon, when the initial rating of the obligor has not changed and the market risk variables equal their expected values, and the actual instrument's value at the risk horizon:

$$L(H) = \sum_{n=1}^{N} L_n(H) = \sum_{n=1}^{N} \left(p_n(\eta_0^n; E^P[X]; P_n) - p_n\left(\eta_H^n; X; P_n\right)\right).$$

Industry standards, such as the well-known credit portfolio models CreditMetrics, CreditPortfolioView, or CreditRisk$^+$, can be seen as special cases of the general modeling approach described above. All of these models have in common that stochastic fluctuations of market risk factors are not considered for the revaluation of the instruments at the risk horizon:

$$p_n\left(\eta_H^n; X; P_n\right) = p_n\left(\eta_H^n; P_n\right) \quad \forall n \in \{1, \ldots, N\}. \tag{4}$$

3.2. A special case: CreditMetrics with integrated correlated interest rate risk

As an example of the general integrated market and credit portfolio model described before, in this section, the usual CreditMetrics framework is extended by interest rate risk, which is correlated with transition risk. This model is used later for the numerical experiments. It is assumed that the return R_n on firm n's assets until the risk horizon can be described by a normally distributed random variable, which is – without loss of generality – standardized:

$$R_n = \sqrt{\rho_R - \rho_{X_r,R}^2} Z + \rho_{X_r,R} X_r + \sqrt{1 - \rho_R} \varepsilon_n \quad \left(\rho_{X_r,R}^2 \leqslant \rho_R, n \in \{1, \ldots, N\}\right), \tag{5}$$

where Z, X_r, and $\varepsilon_1, \ldots, \varepsilon_N$ are mutually independent, standard normally distributed stochastic variables. The variables Z and X_r represent systematic credit risk, by which all firms are affected, whereas ε_n stands for idiosyncratic credit risk of obligor. The specification (5) ensures that the correlation $\mathrm{Corr}(R_n, R_m)$ between the asset returns of two different obligors n and m is equal to ρ_R. From the linear factor model (5) for the firms' asset returns, the conditional transition probabilities (see (1)) can be computed. The risk-free short rate is modeled, for simplicity, as a mean-reverting Ornstein–Uhlenbeck process introduced already by Vasicek (1977). This implies the following representation of the short rate at the risk horizon H:

$$r(H) = \phi + (r(0) - \phi)\mathrm{e}^{-\kappa H} + \sqrt{\frac{\sigma_r^2}{2\kappa}(1 - \mathrm{e}^{-2\kappa H})} X_r \tag{6}$$

where $\kappa, \phi, \sigma_r \in \mathbb{R}_+$. The random variable $X_r \sim N(0,1)$ also enters the definition (5) of the firms' asset returns. Thus, the correlation between the asset returns and the risk-free interest rates is $\rho_{X_r,R}$. The simulation of the possible ratings $1, \ldots, K$ of the obligors at the risk horizon proceeds exactly as described in the technical document of CreditMetrics (see Gupton et al. (1997, p. 85)). The risk horizon H is set equal to 1 year.

Two different portfolio compositions are considered: first, a portfolio of defaultable zero coupon bonds, and, second, a portfolio of European call options on default risk-free zero coupon bonds with counterparty risk. The price of a zero coupon bond at the risk horizon, whose issuer has not defaulted until this time, is calculated by discounting the future cash flow with the risk-adjusted spot yield appropriate for the issuer's rating at the risk horizon. This yield is composed of the stochastic risk-free spot yield (evolving according to the Vasicek model) and a non-stochastic credit spread of the respective rating grade. In the Vasicek model, the stochastic risk-free spot yield can easily be calculated in closed-form and is a linear function of the risk factor X_r appearing in (6). The credit spreads are not modeled as random variables to keep the integrated market and credit portfolio model as simple as possible. However, introducing stochastic credit spreads would be natural extension of the model considered in this paper (see Grundke (2005) for such a model). If an issuer of a zero coupon bond has already defaulted until the risk horizon, the value of the bond is set equal to a constant fraction δ of the value of a risk-free, but otherwise identical, zero coupon bond. The assumption of a constant recovery rate is again for simplicity; the IS technique would also work with (conditionally) independent recovery rates. For the pricing of an European call option on a (default) risk-free zero coupon bond with counterparty risk, it is assumed that in the case of a default, the recovery payment is due at the maturity date T_n^C of the option and that it is an exogenous fraction δ of the option's regular pay off.[6] Furthermore, independence between the movements of the risk-free interest rates and the credit quality changes of the counterparties is assumed for the pricing of the options. With these assumptions, the price of a call written by counterparty n, whose rating at the risk horizon is $\eta_H^n \in \{1, \ldots, K\}$, is given by

[6] Thus, for both instruments, the zero coupon bond as well as the interest rate option with counterparty risk, the recovery payment in the case of a default of the obligor corresponds to the payment a fraction δ of the value of a default risk-free but otherwise identical instrument at the default time.

$$C^*\left(\eta_H^n; X_r; P_n\right) = \delta C(X_r; P_n) + (1-\delta) C(X_r; P_n) \widetilde{P}\left(\tau_n > T_n^C | \eta_H^n\right). \tag{7}$$

Here, τ_n denotes the default time of counterparty n, $\widetilde{P}$ is the risk-neutralized pricing measure, $C(X_r; P_n)$ is the price of an European call option *without* any counterparty risk in the term structure model of Vasicek (1977), and P_n is the vector of relevant option parameters (e.g., exercise price E_n, expiration date T_n^C). Assuming that a default is an absorbing state under $\widetilde{P}$, the event $\{\tau_n > T_n^C\}$ is equivalent to the event $\{\eta^n_{T_n^C} \neq K\}$. The probability of the latter event can simply be calculated by summing up all individual risk-neutral probabilities for a rating change from η_H^n to a non-default state within the time interval $[H, T_n^C]$. Given the assumed independence between the risk-free interest rates and the rating transitions for pricing purposes, the transition probabilities under the risk-neutralized pricing measure $\widetilde{P}$ can easily be inferred from the prices of defaultable bonds issued by the respective counterparty (see, e.g. Jarrow et al. (1997)).

4. Importance sampling techniques for the general approach

When excess probabilities $P(L(H) > y)$ corresponding to large values y are computed with ordinary Monte Carlo estimators

$$P(L(H) > y) = E^P\left[1_{\{L(H) > y\}}\right] \approx \frac{1}{D} \sum_{d=1}^{D} 1_{\{L(H)^{(d)} > y\}}, \tag{8}$$

a large number of simulation runs D is needed to achieve a sufficient accuracy. The basic idea of IS for estimating excess probabilities or percentiles is to shift probability mass to the region of interest around y. This is done by changing the probability distribution of the underlying risk factors. The usage of IS can lead to an improved convergence (in probabilistic terms) of the risk measure estimators when increasing the number of simulation runs. Thus, less simulation runs are necessary to achieve a required accuracy.

In this section, two IS techniques that were developed for a pure credit risk portfolio model and for a pure market risk portfolio model, respectively, are applied to the integrated market and credit portfolio model described in the previous section. Specific problems of this obvious strategy that result from the integrated view of market and credit risk are discussed. Afterwards, in Section 5, the effectiveness of these adapted IS techniques is tested by means of numerical experiments.

4.1. Application of the two-step-IS technique of Glasserman and Li (2005)

4.1.1. Method

In this section, the two-step-IS technique of GL (2005) is applied to the general integrated market and credit portfolio model described in Section 3.1. If we employed a one-step-IS technique and changed only the probability distribution of the systematic risk factors Z and X, the resulting IS estimator of the excess probability $P(L(H) > y)$ is

$$P(L(H) > y) \approx \frac{1}{D} \sum_{d=1}^{D} E^{P^*}\left[1_{\{L(H)^{(d)} > y\}} | Z^{(d)}, X^{(d)}\right] \cdot \frac{f(Z^{(d)}, X^{(d)})}{g(Z^{(d)}, X^{(d)})}. \tag{9}$$

In (9), $Z^{(d)}$ and $X^{(d)} (d \in \{1, \ldots, D\})$ are realizations of the systematic risk factors under the IS distribution P^* with density function $g(Z, X)$. $f(Z, X)$ is the density function of the systematic risk factors under the original distribution P. The fraction in (9) that contains both density functions is the so-called likelihood ratio, which reverses the transformation of the original probability measure. For computing the remaining conditional expectation in (9), the individual rating transitions *could* be sampled according to the original conditional transition probabilities (1). However, in a second step, these conditional transition probabilities are also modified to make larger losses more probable:

$$h_{n,i,k}(Z, X) := \frac{\mathrm{e}^{\theta L_{n,i,k}(X,H)} f_{n,i,k}(Z, X)}{E^{P^*}\left[\mathrm{e}^{\theta L_{n,i,\eta_H^n}(X,H)} | Z, X\right]} \quad (1 \leqslant i \leqslant K-1; 1 \leqslant k \leqslant K), \tag{10}$$

where $L_{n,i,k}(X, H)$ is the loss of obligor n's instrument after a migration from rating grade i to k within the risk horizon, and θ is a non-negative parameter. For $\theta > 0$ and $L_{n,i,k}(X, H) > 0$, the transition probabilities are increased, whereas for $\theta > 0$ and $L_{n,i,k}(X, H) < 0$, they are diminished. The absolute increase or decrease of the probabilities is larger, the larger the individual losses or gains of obligor n's instrument incurred by the respective rating change are. For $\theta = 0$, the original transition probabilities are not altered. Hence, in general, the downgrade probabilities are increased and the upgrade probabilities are decreased. However, due to the integration of market risk, it is also possible that, for example, a downgrade probability is decreased. This would happen in a specific scenario in which a decrease of an instrument's value caused by a downgrade of the issuer is overcompensated by a value increase caused by a movement in the market risk factors. The definition (10) of the conditional IS transition probabilities corresponds to a so-called exponential twist of the original conditional transition probabilities. This ensures that the conditional likelihood ratio has a nice representation: it only depends on the parameter θ, the portfolio loss $L(H)$, and the cumulant generating function $\psi_{L(H)|Z,X}(\theta)$, which is the logarithm of the moment generating function of the portfolio loss variable $L(H)$. Thus, this yields the following representation of the conditional expectation in (9):

$$E^{P^*}\left[1_{\{L(H)>y\}}|Z,X\right] = E^{\widetilde{P}_\theta}\left[1_{\{L(H)>y\}}\mathrm{e}^{-\theta L(H)+\psi_{L(H)|Z,X}(\theta)}|Z,X\right], \tag{11}$$

where

$$\psi_{L(H)|Z,X}(\theta) = \ln\left(E^{P^*}\left[\mathrm{e}^{\theta L(H)}|Z,X\right]\right). \tag{12}$$

Here, $E^{\widetilde{P}_\theta}[\cdot]$ is the expectation operator under the IS probability measure $\widetilde{P}_\theta$ for the conditional transitions, which depends on the parameter $\theta \in \mathbb{R}_+$.

The two essential problems which remain are finding an effective IS distribution for the systematic risk factors and for the conditional rating transitions. Starting with the latter problem, the parameter θ has to be chosen in such a way that the variance or, equivalently, the second moment of the estimator under the IS distribution is minimized. As computing the second moment under the – up to now unknown – IS distribution is difficult, instead, an upper boundary of the second moment is minimized:

$$E^{\widetilde{P}_\theta}\left[\left(1_{\{L(H)>y\}}\mathrm{e}^{-\theta L(H)+\psi_{L(H)|Z,X}(\theta)}\right)^2|Z,X\right] \underset{\theta\geqslant 0}{\leqslant} \mathrm{e}^{-2(\theta y-\psi_{L(H)|Z,X}(\theta))}. \tag{13}$$

Then, the in this sense optimal parameter θ is given by the solution of the following optimization problem:

$$\theta_y(Z,X) = \begin{cases} \text{unique solution to } \frac{\partial}{\partial\theta}\left(\psi_{L(H)|Z,X}(\theta)\right) = y & \text{for } y > \frac{\partial}{\partial\theta}\left(\psi_{L(H)|Z,X}(\theta)\right)\Big|_{\theta=0}, \\ 0 & \text{for } y \leqslant \frac{\partial}{\partial\theta}\left(\psi_{L(H)|Z,X}(\theta)\right)\Big|_{\theta=0}. \end{cases} \tag{14}$$

Here, the twisting parameters are restricted to non-negative values because otherwise the probability for transitions that yield a positive loss could be reduced. The parameter $\theta_y(Z,X)$ is unique because the (conditional) cumulant generating function $\psi_{L(H)|Z,X}(\theta)$ is strictly convex in θ. Choosing the parameter $\theta_y(Z,X)$ according to (14) ensures that conditional credit portfolio losses around y are no longer rare events under the IS distribution $\widetilde{P}_{\theta_y(Z,X)}$ for the conditional rating transitions. This can be seen from the following relation (see Glasserman (2004, p. 261)):

$$E^{\widetilde{P}_{\theta_y(Z,X)}}[L(H)|Z,X] = \frac{\partial}{\partial\theta}\left(\psi_{L(H)|Z,X}(\theta)\right)\Big|_{\theta=\theta_y(Z,X)} \underset{(14),}{=} y. \tag{15}$$

uppercase

Next, an IS density function for the systematic risk factors Z and X has to be found. In this situation, the optimal, this means zero-variance, IS density function for the systematic risk factors can be shown to be proportional to the product of the conditional excess probability and the original density function of the risk factors (see Glasserman (2004, p. 256)). However, sampling from this density is generally not possible because the normalization constant required to make it a density is just the probability $P(L(H)>y)$ that is looked for. For this problem, the following solution has been suggested: choose as IS distribution a normal density with the same mode as the optimal density, which is given by the solution of the following optimization problem:

$$\begin{aligned} \mu^{\mathrm{IS}}(y) &= (\mu_1^Z,\ldots,\mu_C^Z,\mu_1^X,\ldots,\mu_M^X)^{\mathrm{T}} \\ &= \underset{z_1,\ldots,z_C,x_1,\ldots,x_M\in\mathbb{R}}{\arg\max} E^{\widetilde{P}_{\theta_y(z,x)}}\left[1_{\{L(H)>y\}}\mathrm{e}^{-\theta_y(z,x)L(H)+\psi_{L(H)|Z=z,X=x}(\theta_y(z,x))}|Z=z,X=x\right]\cdot\mathrm{e}^{-0.5\left(\sum_{c=1}^C z_c^2+\sum_{m=1}^M x_m^2\right)}. \end{aligned} \tag{16}$$

In (16), the IS representation for the conditional excess probability $P(L(H)>y|Z,X)$ has been inserted. Furthermore, it has been assumed that the systematic risk factors are uncorrelated, standard normally distributed random variables under the original probability distribution P.[7] To simplify the above optimization problem, again an upper boundary of the conditional expectation is used. This yields the following related optimization problem[8]:

[7] If the joint distribution of the random vector (Z,X) is a multivariate normal distribution, this assumption is without loss of generality because a set of correlated normally distributed random variables can always be represented by a linear combination of orthogonal standard normally distributed random variables.

[8] See Glasserman et al. (2007) for problems when determining the IS means of the systematic risk factors in a multifactor credit portfolio model this way and for a possible solution.

$$\mu^{\mathrm{IS}}(y) \approx \underset{z_1,\ldots,z_C,x_1,\ldots,x_M\in\mathbb{R}}{\arg\max} -\theta_y(z,x)y + \psi_{L(H)|Z=z,X=x}(\theta_y(z,x)) - 0.5\left(\sum_{c=1}^{C} z_c^2 + \sum_{m=1}^{M} x_m^2\right). \tag{17}$$

The IS distribution for the systematic risk factors Z and X is a multivariate normal distribution with mean vector given by (17) and covariance matrix equal to the identity matrix I. In the end, this yields the following two-step-IS estimator for the excess probabilities:

$$P(L(H) > y) \approx \frac{1}{D}\sum_{d=1}^{D} 1_{\{L(H)^{(d)}>y\}} \cdot l_1^y(Z^{(d)},X^{(d)}) \cdot l_2(Z^{(d)},X^{(d)}), \tag{18}$$

where

$$l_1^y(Z^{(d)},X^{(d)}) := \mathrm{e}^{-\theta_y(Z^{(d)},X^{(d)})L(H)^{(d)}+\psi_{L(H)|Z^{(d)},X^{(d)}}(\theta_y(Z^{(d)},X^{(d)}))} \tag{19}$$

and

$$l_2(Z^{(d)},X^{(d)}) := \mathrm{e}^{-\sum_{c=1}^{C}\left(Z_c^{(d)}\mu_c^Z-0.5(\mu_c^Z)^2\right)-\sum_{m=1}^{M}\left(X_m^{(d)}\mu_m^X-0.5(\mu_m^X)^2\right)}. \tag{20}$$

The first exponential term is the likelihood ratio for the conditional IS transition probabilities, and the second exponential term is the likelihood ratio for the IS distribution of the systematic risk factors. The systematic risk factors are sampled according to $N((\mu_1^Z,\ldots,\mu_C^Z,\mu_1^X,\ldots,\mu_M^X)^{\mathrm{T}},I)$ and – conditional on (Z,X) – the portfolio loss $L(H)$ is sampled according to the IS transition probabilities $h_{n,i,k}(Z,X)$ (see (10)).

4.1.2. Discussion

GL (2005) apply the above two-step-IS procedure to a CreditMetrics-style, pure default mode model. In their modeling approach, the (conditional) cumulant generating function and its derivative with respect to θ, respectively, are given by

$$\psi_{L(H)|Z}(\theta) = \sum_{n=1}^{N} \ln\left(1 + f_{n,\eta_0^n,K}(Z)(\mathrm{e}^{\theta c_n}-1)\right) \tag{21}$$

and

$$\frac{\partial}{\partial\theta}\left(\psi_{L(H)|Z}(\theta)\right) = \sum_{n=1}^{N} \frac{f_{n,\eta_0^n,K}(Z)c_n\mathrm{e}^{\theta c_n}}{1+f_{n,\eta_0^n,K}(Z)(\mathrm{e}^{\theta c_n}-1)}. \tag{22}$$

Here, $f_{n,\eta_0^n,K}(Z)$ is the conditional default probability of obligor n, and c_n is the loss incurred by a default of this obligor. As

$$\frac{\partial}{\partial\theta}\left(\psi_{L(H)|Z}(\theta)\right)\Big|_{\theta=0} = E^P[L(H)|Z] \tag{23}$$

and

$$\lim_{\theta\to\infty}\frac{\partial}{\partial\theta}\left(\psi_{L(H)|Z}(\theta)\right) = \sum_{n=1}^{N} c_n, \tag{24}$$

the optimization problem (14) has a solution $\theta \in (0,\infty)$ for all values y larger than the conditional expected loss (23) under the original transition probabilities and smaller than the maximum loss (24).

In the extended market and credit portfolio model considered in this paper, the potential loss is itself a random variable, which depends on the realization of the market risk factors. Defining the conditional loss of obligor n's instrument, given that the obligor's rating at the risk horizon is k, as

$$L_{n,k}(H)|X := p_n(\eta_0^n;E^P[X];P_n) - p_n(k;X;P_n), \tag{25}$$

the optimization problem (14) has a solution $\theta \in (0,\infty)$ for all values

$$y \in \left(\underbrace{\textstyle\sum_{n=1}^{N} E^P\left[L_{n,\eta_H^n}(H)|Z,X\right]}_{\text{conditional expected loss}},\ \underbrace{\textstyle\sum_{n=1}^{N}\max_{1\leqslant k\leqslant K} L_{n,k}(H)|X}_{\text{conditional maximum loss}}\right). \tag{26}$$

As

$$\psi_{L(H)|Z,X}(\theta)=\sum_{n=1}^{N}\ln\left(\sum_{k=1}^{K}\mathrm{e}^{\theta L_{n,k}(H)|X}f_{n,\eta_0^n,k}(Z)\right) \tag{27}$$

and hence

$$\frac{\partial}{\partial\theta}\left(\psi_{L(H)|Z,X}(\theta)\right)=\sum_{n=1}^{N}\frac{\sum_{k=1}^{K}\mathrm{e}^{\theta L_{n,k}(H)|X}f_{n,\eta_0^n,k}(Z)L_{n,k}(H)|X}{\sum_{k=1}^{K}\mathrm{e}^{\theta L_{n,k}(H)|X}f_{n,\eta_0^n,k}(Z)}, \tag{28}$$

the lower boundary in (26) follows directly from setting $\theta=0$ in (28). Without loss of generality, $L_{n,K}(H)|X=\max_{1\leqslant k\leqslant K}L_{n,k}(H)|X$ can be assumed for all $n\in\{1,\dots,N\}$. This assumption implies that the maximum loss is always incurred by a default. Then, the upper boundary in (26) can be seen from:

$$\begin{aligned}\frac{\partial}{\partial\theta}\left(\psi_{L(H)|Z,X}(\theta)\right)&=\sum_{n=1}^{N}\left(\frac{\sum_{k=1}^{K}\mathrm{e}^{\theta L_{n,k}(H)|X}f_{n,\eta_0^n,k}(Z)L_{n,k}(H)|X}{\sum_{k=1}^{K}\mathrm{e}^{\theta L_{n,k}(H)|X}f_{n,\eta_0^n,k}(Z)}\cdot\frac{\mathrm{e}^{-\theta L_{n,K}(H)|X}}{\mathrm{e}^{-\theta L_{n,K}(H)|X}}\right)\\&=\sum_{n=1}^{N}\left(\frac{\sum_{k=1}^{K-1}\mathrm{e}^{\theta\left(L_{n,k}(H)|X-L_{n,K}(H)|X\right)}f_{n,\eta_0^n,k}(Z)L_{n,k}(H)|X+f_{n,\eta_0^n,K}(Z)L_{n,K}(H)|X}{\sum_{k=1}^{K-1}\mathrm{e}^{\theta\left(L_{n,k}(H)|X-L_{n,K}(H)|X\right)}f_{n,\eta_0^n,k}(Z)+f_{n,\eta_0^n,K}(Z)}\right).\end{aligned} \tag{29}$$

Due to $L_{n,k}(H)|X-L_{n,K}(H)|X<0$ for all $k\in\{1,\dots,K-1\}$, (29) converges to the upper boundary in (26) for $\theta\to\infty$.

However, depending on the instrument type and the realization of the market risk factors X, the interval (26) can be rather small. This problem occurs when losses are mainly caused by changes in the market risk factors and not by rating transitions. This is true, for example, for portfolios of options, which are considered in the numerical experiments of Section 5. The consequence is that the optimization problem (14) might not be well defined. If $y>\sum_{n=1}^{N}\max_{1\leqslant k\leqslant K}L_{n,k}(H)\mid X^{(d)}$ for a specific simulation run $(Z^{(d)},X^{(d)})$, this yields $\theta=\infty$ in (14). In this case, the likelihood ratio of the conditional transition probabilities converges with probability one to the product of the conditional default probabilities:

$$\exp\left(-\theta_y(Z^{(d)},X^{(d)})L(H)^{(d)}+\psi_{L(H)|Z^{(d)},X^{(d)}}(\theta_y(Z^{(d)},X^{(d)}))\right)\underset{\theta_y(Z^{(d)},X^{(d)})\to\infty}{\longrightarrow}\prod_{n=1}^{N}f_{n,\eta_0^n,K}(Z^{(d)}). \tag{30}$$

Here, it is again assumed $L_{n,K}(H)|X=\max_{1\leqslant k\leqslant K}L_{n,k}(H)|X$ for all $n\in\{1,\dots,N\}$. The limit result (30) can be seen as follows. For the conditional likelihood ratio, the following representation can be derived:

$$\begin{aligned}&\exp\left(-\theta_y(Z^{(d)},X^{(d)})L(H)^{(d)}+\psi_{L(H)|Z^{(d)},X^{(d)}}(\theta_y(Z^{(d)},X^{(d)}))\right)\\&=\exp\left(\sum_{n=1}^{N}\left(-\theta_y(Z^{(d)},X^{(d)})\right)L_{n,\eta_H^n}(H)|X^{(d)}+\sum_{n=1}^{N}\ln\left(\sum_{k=1}^{K}\mathrm{e}^{\theta_y(Z^{(d)},X^{(d)})L_{n,k}(H)|X^{(d)}}f_{n,\eta_0^n,k}(Z^{(d)})\right)\right)\\&=\prod_{n=1}^{N}\exp\left(\left(-\theta_y(Z^{(d)},X^{(d)})\right)L_{n,\eta_H^n}(H)|X^{(d)}+\ln\left(\sum_{k=1}^{K}\mathrm{e}^{\theta_y(Z^{(d)},X^{(d)})L_{n,k}(H)|X^{(d)}}f_{n,\eta_0^n,k}(Z^{(d)})\right)\right)\\&=\prod_{n=1}^{N}\left(\sum_{k=1}^{K}\exp\left(\theta_y(Z^{(d)},X^{(d)})\left(L_{n,k}(H)|X^{(d)}-L_{n,\eta_H^n}(H)|X^{(d)}\right)\right)f_{n,\eta_0^n,k}(Z^{(d)})\right).\end{aligned} \tag{31}$$

From the definition (10) of the conditional transition probabilities under the IS probability measure $\widetilde{P}_{\theta_y(Z^{(d)},X^{(d)})}$ follows that all non-default probabilities converge to zero for $\theta_y(Z^{(d)},X^{(d)})\to\infty$, whereas the default probability goes to one:

$$\begin{aligned}h_{n,\eta_0^n,k}(Z^{(d)},X^{(d)})&=\frac{\mathrm{e}^{\theta_y(Z^{(d)},X^{(d)})L_{n,k}(H)|X^{(d)}}f_{n,\eta_0^n,k}(Z^{(d)})}{\sum_{s=1}^{K}\mathrm{e}^{\theta_y(Z^{(d)},X^{(d)})L_{n,s}(H)|X^{(d)}}f_{n,\eta_0^n,s}(Z^{(d)})}\cdot\frac{\mathrm{e}^{-\theta_y(Z^{(d)},X^{(d)})L_{n,K}(H)|X^{(d)}}}{\mathrm{e}^{-\theta_y(Z^{(d)},X^{(d)})L_{n,K}(H)|X^{(d)}}}\\&=\frac{\mathrm{e}^{\theta_y(Z^{(d)},X^{(d)})\left(L_{n,k}(H)|X^{(d)}-L_{n,K}(H)|X^{(d)}\right)}f_{n,\eta_0^n,k}(Z^{(d)})}{\sum_{s=1}^{K-1}\mathrm{e}^{\theta_y(Z^{(d)},X^{(d)})\left(L_{n,s}(H)|X^{(d)}-L_{n,K}(H)|X^{(d)}\right)}f_{n,\eta_0^n,s}(Z^{(d)})+f_{n,\eta_0^n,K}(Z^{(d)})}\underset{\theta_y(Z^{(d)},X^{(d)})\to\infty}{\longrightarrow}\begin{cases}0 & \text{for } k\in\{1,\dots,K-1\},\\1 & \text{for } k=K.\end{cases}\end{aligned}$$

Thus, in the limit, this yields $L_{n,\eta_H^n}(H)\mid X^{(d)}=L_{n,K}(H)\mid X^{(d)}$ with probability one in (31). This implies

$$L_{n,k}(H)|X^{(d)}-L_{n,\eta_H^n}(H)|X^{(d)}<0$$

for all $k\in\{1,\dots,K-1\}$ and

$$L_{n,K}(H)|X^{(d)}-L_{n,\eta_H^n}(H)|X^{(d)}=0$$

with probability one, from which (30) results. As the default probabilities $f_{n,\eta_0^n,K}(Z^{(d)})$ are usually very small, (30) will already be indistinguishable from zero for the computer for a moderate number N of obligors. The consequence is that this simulation run has no contribution to the probability estimator (18). In the CreditMetrics-style, pure default mode model, which GL (2005) employ for their numerical examples, this problem cannot occur because no initial percentile guess y should be larger than the non-stochastic maximum potential loss $\sum_{n=1}^{N} c_n$.

If $y < \sum_{n=1}^{N} E^P[L_{n,\eta_H^n}(H) \mid Z, X]$, the parameter θ is set equal to zero (see (14)). This just implies that there is no IS for the conditional transition probabilities and that the conditional likelihood ratio is one. However, even if y is in the interval (26), the parameter θ fulfilling $\frac{\partial}{\partial\theta}(\psi_{L(H)|Z,X}(\theta)) = y$ can be very large (see the example in Section 5.3 in the following). This can cause an overflow problem when calculating the conditional cumulant generating function and its derivative.

4.2. Application of the market risk portfolio IS technique

4.2.1. Method

In this section, it is tried to combine an IS technique originally developed for pure market risk portfolio models with the previous two-step-IS technique originally developed for pure credit risk portfolio models. This idea might suggest itself because we deal here with integrated *market and credit* portfolio models. Of course, the hypothesis is that this combined IS technique is even more effective than the two-step-IS approach.

Instead of considering the optimization problem (17) for finding the IS means of both kinds of systematic risk factors, now, this procedure is only carried out for the systematic credit risk factors $Z_1, \ldots, Z_C$, whereas the IS distribution for the market risk factors $X_1, \ldots, X_M$ is determined in an intermediate step. For finding the IS distribution of the market risk factors, it is assumed that the credit portfolio is default risk-free and that all obligors remain in their initial rating class until the risk horizon. With this assumption, the approach of Glasserman et al. (2000) developed for pure market risk portfolio models can be applied. Their method employs a delta–gamma approximation of the portfolio loss at the risk horizon for selecting a variance reducing IS distribution of the market risk factors. Here, as in the previous section, it is assumed that the market risk factors are uncorrelated and standard normally distributed.[9] The random variable which represents the credit portfolio loss that is only due to movements in the market risk factors over the risk horizon is defined as

$$L^{\mathrm{wtr}}(X,H) = \sum_{n=1}^{N} L_n^{\mathrm{wtr}}(X,H) = \sum_{n=1}^{N} \left(p_n(\eta_0^n; E^P[X]; P_n) - p_n(\eta_0^n; X; P_n)\right), \tag{32}$$

where the upper index wtr indicates that this is the loss *w*ithout *t*ransition *r*isk. Note that both terms in the sum in (32) depend on the current rating η_0^n. For this random variable, a specific quadratic approximation, the so-called delta–gamma approximation, is introduced:

$$L^{\mathrm{wtr}}(X,H) \approx L^{\mathrm{wtr},\Delta,\Gamma}(X,H) = \underbrace{L^{\mathrm{wtr}}(E^P[X],H)}_{=0} + \delta^{\mathrm{T}}(X - \underbrace{E^P[X]}_{=0}) + 0.5(X - \underbrace{E^P[X]}_{=0})^{\mathrm{T}}\Gamma(X - \underbrace{E^P[X]}_{=0}), \tag{33}$$

where the column vector $\delta = (\delta_m)_{1 \leqslant m \leqslant M}$ contains the first derivatives of $L^{\mathrm{wtr}}(X,H)$ with respect to the market risk factors, and the matrix $\Gamma = (\Gamma_{m,l})_{1 \leqslant m,l \leqslant M}$ is the Hessian matrix with the second derivatives of $L^{\mathrm{wtr}}(X,H)$ with respect to the market risk factors. Hence, (33) is just a second-order Taylor series expansion of the credit portfolio loss $L^{\mathrm{wtr}}(X,H)$ around the expected value of the market risk factors at the risk horizon. After diagonalization of the Hessian matrix Γ, the following more convenient expression for the quadratic approximation (33) can be derived:

$$L^{\mathrm{wtr}}(X,H) \approx L^{\mathrm{wtr},\Delta,\Gamma}(X,H) = \sum_{m=1}^{M} \left(b_m S_m + \lambda_m S_m^2\right) = Q(S), \tag{34}$$

where $S = U^{\mathrm{T}}X \sim N(0,I)$, U is an orthogonal matrix whose columns are the eigenvectors of 0.5Γ, λ_m ($m \in \{1,\ldots,M\}$) are the components of the diagonal matrix Λ containing the eigenvalues of $0.5\Gamma = U\Lambda U^{\mathrm{T}}$, and b_m ($m \in \{1,\ldots,M\}$) are the components of the vector $\delta^{\mathrm{T}}U$. In the next step, the approximation $L^{\mathrm{wtr}}(X,H) \approx Q(S)$ is used for finding an IS distribution for the transformed market risk factors S under which large values of the portfolio loss $L^{\mathrm{wtr}}(X,H)$ are generated with an higher probability than under the original distribution. As IS distribution, a multivariate normal distribution with means

$$\mu_m(\theta^{\mathrm{wtr}}) = \frac{\theta^{\mathrm{wtr}} b_m}{1 - 2\theta^{\mathrm{wtr}}\lambda_m} \quad (m \in \{1,\ldots,M\}), \tag{35}$$

[9] For the necessary transformations when the market risk factors are multivariate normally distributed (in particular correlated), see Glasserman (2004, p. 486) and Glasserman et al. (2000, p. 1351). For lognormally or t-distributed risk factors, see Glasserman et al. (2000, p. 1351) and Glasserman et al. (2002).

and covariances

$$\sigma^2_{ml}(\theta^{\mathrm{wtr}}) = \begin{cases} \frac{1}{1-2\theta^{\mathrm{wtr}}\lambda_m} & (m=l) \\ 0 & (m\neq l) \end{cases} \quad (m,l \in \{1,\ldots,M\}) \tag{36}$$

is chosen. For the parameter θ^{wtr}, it is required that $0 \leqslant \theta^{\mathrm{wtr}} < (\max_{1\leqslant m\leqslant M} 2\lambda_m)^{-1}$ if $\max_{1\leqslant m\leqslant M}\lambda_m > 0$. Choosing the IS means and variances as defined in (35) and (36) ensures that those random variables S_m for which $b_m > 0$ ($b_m < 0$) have a positive (negative) mean and that those S_m for which $\lambda_m > 0$ have a large variance. This in turn makes larger values of $Q(S)$ and hence larger portfolio losses $L^{\mathrm{wtr}}(X,H)$ more likely. A further consequence of the choice (35) and (36) for the entries of the mean vector $\mu(\theta^{\mathrm{wtr}})$ and the covariance matrix $\Sigma(\theta^{\mathrm{wtr}})$, respectively, is the following simple likelihood ratio, which is typical for exponential twisting:

$$l_3(S) = \frac{\mathrm{e}^{-\frac{1}{2}S^{\mathrm{T}}S}}{\det(\Sigma(\theta^{\mathrm{wtr}}))^{-\frac{1}{2}}\mathrm{e}^{-\frac{1}{2}(S-\mu(\theta^{\mathrm{wtr}}))^{\mathrm{T}}\Sigma(\theta^{\mathrm{wtr}})^{-1}(S-\mu(\theta^{\mathrm{wtr}}))}} = \mathrm{e}^{-\theta^{\mathrm{wtr}}Q(S)+\psi_Q(\theta^{\mathrm{wtr}})}, \tag{37}$$

where $\psi_Q(\theta^{\mathrm{wtr}}) = \ln(E^P[\mathrm{e}^{\theta^{\mathrm{wtr}}Q(S)}])$ is the cumulant generating function of the random variable $Q(S)$. In a final step, the twisting parameter θ^{wtr} has to be determined. For this, the approximation $P(L^{\mathrm{wtr}}(H) > y^*) \approx P(Q(S) > y^*)$ is used and a parameter θ^{wtr}, which is effective for estimating the probability on the right-hand side, is computed, hoping that it is also effective for estimating the probability on the left-hand side.[10] Again, the parameter θ^{wtr} is determined by minimizing an upper boundary of the second moment of the excess probability estimator under the IS distribution:

$$E^{\widetilde{P}_{\theta^{\mathrm{wtr}}}}\left[\left(1_{\{Q(S)>y^*\}}l_3(S)\right)^2\right] \underset{\theta^{\mathrm{wtr}}\geqslant 0}{\leqslant} \mathrm{e}^{-2\left(\theta^{\mathrm{wtr}}y^*-\psi_Q(\theta^{\mathrm{wtr}})\right)}. \tag{38}$$

This yields

$$\theta^{\mathrm{wtr}}_{y^*} = \begin{cases} \text{unique solution to } \frac{\partial}{\partial\theta^{\mathrm{wtr}}}\left(\psi_Q(\theta^{\mathrm{wtr}})\right) = y^* & \text{for } y^* > \frac{\partial}{\partial\theta^{\mathrm{wtr}}}\left(\psi_Q(\theta^{\mathrm{wtr}})\right)\big|_{\theta^{\mathrm{wtr}}=0}, \\ 0 & \text{for } y^* \leqslant \frac{\partial}{\partial\theta^{\mathrm{wtr}}}\left(\psi_Q(\theta^{\mathrm{wtr}})\right)\big|_{\theta^{\mathrm{wtr}}=0}. \end{cases} \tag{39}$$

As

$$\frac{\partial}{\partial\theta^{\mathrm{wtr}}}\left(\psi_Q(\theta^{\mathrm{wtr}})\right)\bigg|_{\theta^{\mathrm{wtr}}=0} = \sum_{m=1}^{M}\lambda_m \tag{40}$$

and

$$\frac{\partial}{\partial\theta^{\mathrm{wtr}}}\left(\psi_Q(\theta^{\mathrm{wtr}})\right) \underset{\theta^{\mathrm{wtr}}\to\infty}{\to} -\sum_{m=1}^{M}\frac{b_m^2}{4\lambda_m}, \tag{41}$$

the above optimization problem (39) is well defined and has a non-zero solution for initial guesses y^* out of the interval

$$y^* \in \left(\sum_{m=1}^{M}\lambda_m, -\sum_{m=1}^{M}\frac{b_m^2}{4\lambda_m}\right). \tag{42}$$

However, due to the additional restriction for θ^{wtr} mentioned after (36), which ensures that the variance of the market risk factors under the IS distribution is non-negative, the admissible interval for y^* might even have a smaller upper boundary.

4.2.2. Discussion

Next, the problem arises how to combine the two-step-IS technique for the systematic credit risk factors Z with the IS technique developed for pure market risk portfolio models. There are several possibilities how to do this, which are discussed in the following. As the excess probability $P(L(H) > y)$ can be represented by

$$\begin{aligned} P(L(H) > y) &= E^P[P(L(H) > y|Z,X)] = E^P\left[E^{\widetilde{P}_{\theta_y(Z,US)}}\left[1_{\{L(H)>y\}}l_1(Z,US)|Z,S\right]\right] \\ &= E^P\left[E^P\left[E^{\widetilde{P}_{\theta_y(Z,US)}}\left[1_{\{L(H)>y\}}l_1(Z,US)|Z,S\right]\Big|Z\right]\right] = E^P\left[E^{\widetilde{P}_{\theta^{\mathrm{wtr}}_{y^*}}}\left[E^{\widetilde{P}_{\theta_y(Z,US)}}\left[1_{\{L(H)>y\}}l_1(Z,US)|Z,S\right]l_3(S)\Big|Z\right]\right], \end{aligned}$$

this yields the following three-step-IS estimator for the excess probability $P(L(H) > y)$:

[10] In the following, it is differed between the two figures y and y^*: y is the initial guess of the percentile of the credit portfolio loss distribution which is looked for, whereas y^* is the initial guess of a percentile of the loss distribution when only market risk, but no transition risk, is considered. Especially for portfolios with a low credit quality, these figures differ significantly, even if the percentiles correspond to the same confidence level.

$$P(L(H) > y) \approx \frac{1}{D}\sum_{d=1}^{D} 1_{\{L(H)^{(d)}>y\}} \cdot l_1(Z^{(d)}, US^{(d)}) \cdot l_3(S^{(d)}) \cdot l_2(Z^{(d)}). \tag{43}$$

Here, the likelihood ratio $l_1(Z^{(d)}, US^{(d)})$ for the conditional transition probabilities is defined as in (19), the likelihood ratio $l_3(S^{(d)})$ for the market risk factors S is given by (37), and the likelihood ratio $l_2(Z^{(d)})$ for the systematic credit risk factors Z is defined analogously to (20). Conditional on the realization of (Z,S), the credit portfolio loss $L(H)$ is sampled according to the IS transition probabilities $h_{n,i,k}(Z, US)$, the vector of transformed market risk factors S is sampled according to (35) and (36), and the vector of systematic credit risk factors Z is sampled according to $N(\mu^{\mathrm{IS}}(y), I)$. Employing an upper boundary for $E^{\widetilde{P}_{\theta_y(Z,US)}}[1_{\{L(H)>y\}} l_1(Z, US) \mid Z, S]$ analogously to (13), the C-dimensional IS mean vector $\mu^{\mathrm{IS}}(y)$ for the systematic credit risk factors Z is given by

$$\mu^{\mathrm{IS}}(y) = \underset{z_1,\ldots,z_C\in\mathbb{R}}{\arg\max}\, E^{\widetilde{P}_{\theta^{\mathrm{wtr}}_{y^*}}}\left[\mathrm{e}^{-\theta_y(z,US)y+\psi_{L(H)|z,US}(\theta_y(z,US))} l_3(S)|Z=z\right]\mathrm{e}^{-0.5\sum_{c=1}^{C} z_c^2}. \tag{44}$$

Solving the optimization problem (44) would be rather involved because there are usually many market risk factors S that are relevant for the value of a portfolio. Thus, a multi-dimensional integral would have to be solved numerically many times in this optimization problem. To circumvent this drawback, an alternative might be to change the order in which the conditional expectations are computed. This yields

$$\begin{aligned} P(L(H) > y) &= E^P[P(L(H) > y|Z,X)] = E^P\left[E^P\left[E^{\widetilde{P}_{\theta_y(Z,US)}}\left[1_{\{L(H)>y\}} l_1(Z, US)|Z,S\right]\Big|S\right]\right] \\ &= E^{\widetilde{P}_{\theta^{\mathrm{wtr}}_{y^*}}}\left[E^P\left[E^{\widetilde{P}_{\theta_y(Z,US)}}\left[1_{\{L(H)>y\}} l_1(Z, US)|Z,S\right]\Big|S\right] l_3(S)\right] \\ &= E^{\widetilde{P}_{\theta^{\mathrm{wtr}}_{y^*}}}\left[E^{P_{\mu_y^{\mathrm{IS}}(S)}}\left[E^{\widetilde{P}_{\theta_y(Z,US)}}\left[1_{\{L(H)>y\}} l_1(Z, US)|Z,S\right] l_2(Z)\Big|S\right] l_3(S)\right], \end{aligned}$$

where the likelihood ratio

$$l_2(Z)|S = \mathrm{e}^{-\sum_{c=1}^{C}\left(Z_c\mu_c^Z(S)-0.5(\mu_c^Z(S))^2\right)} \tag{45}$$

for the systematic risk factors Z as well as their IS means $\mu_y^{\mathrm{IS}}(S) = (\mu_1^Z(S), \ldots, \mu_C^Z(S))^{\mathrm{T}}$

$$\mu_y^{\mathrm{IS}}(S) = \underset{z_1,\ldots,z_C\in\mathbb{R}}{\arg\max}\left(-\theta_y(z, US)y + \psi_{L(H)|z,US}(\theta_y(z, US)) - 0.5\sum_{c=1}^{C} z_c^2\right) \tag{46}$$

now depend on the realization of the market risk factors S. Unfortunately, this optimization problem has the serious disadvantage that it has to be solved for each scenario of the market risk factors S, which makes this approach also computational expensive. A third possibility to combine all three steps, which avoids the computational difficulties of the two previous approaches, is to assume that the market risk factors S equal their IS means $E^{\widetilde{P}_{\theta^{\mathrm{wtr}}_{y^*}}}[S] = \mu(\theta^{\mathrm{wtr}}_{y^*})$ while determining the IS means of the systematic credit risk factors Z. This yields the following optimization problem (instead of (44) and (46)):

$$\mu^{\mathrm{IS}}(y) = \underset{z_1,\ldots,z_C\in\mathbb{R}}{\arg\max}\left(-\theta_y(z, U\mu(\theta^{\mathrm{wtr}}_{y^*}))y + \psi_{L(H)|z,U\mu(\theta^{\mathrm{wtr}}_{y^*})}(\theta_y(z, U\mu(\theta^{\mathrm{wtr}}_{y^*}))) - 0.5\sum_{c=1}^{C} z_c^2\right). \tag{47}$$

The effectiveness of this simplified third approach is tested within the numerical example in the next section.

5. Numerical results

Next, the effectiveness of the IS techniques presented in the previous section is analyzed by means of numerical experiments. For this, the simple example of an integrated market and credit portfolio model as described in Section 3.2 is employed. For a one-factor CreditMetrics-style, default mode model applied to a homogeneous portfolio, GL (2005) are able to derive various theoretical results (mainly concerning asymptotic optimality) about the two-step-IS technique they propose. However, even for the simple integrated market and credit portfolio model that is employed in this section, it seems not possible to derive similar results.[11] That is the reason why only numerical experiments are carried out for demonstrating the performance of the different IS techniques when applied to integrated market and credit portfolio models.

[11] See also Glasserman et al. (2007, p. 6).

5.1. Parameters

First, it is assumed that the credit portfolio consists of $N = 500$ defaultable zero coupon bonds. These are issued by N different obligors, but are otherwise identical. The face value is chosen to be $F = 1$ and the maturity date is $T = 3$, implying a remaining time to maturity of two years at the risk horizon. The simulations are done for the initial ratings $\eta_0 \in \{\mathrm{Aa}, \mathrm{Baa}, \mathrm{B}\}$. As typical parameters for the Vasicek term structure model, $\kappa = 0.4$ and $\sigma_r = 0.01$ are chosen. The mean level ϕ and the initial short rate $r(0)$ are set equal to 0.06. As market price of interest rate risk λ, a value of 0.5 is taken.[12] The recovery rate is set equal to 53.80%. This is the mean of the recovery rate of senior unsecured bonds during 1970 to 1995 reported by Moody's.[13] The employed transition matrix is also from Moody's.[14] The correlation ρ_R between the asset returns is chosen as 0.1 and 0.4, respectively. The former value is within the range of values proposed by the Basle Committee on Banking Supervision for corporate exposures in the IRB approach (see Basel Committee on Banking Supervision (2005)). The latter value is taken to test the effect of extreme asset return correlations. The parameter $\rho_{X_r,R}$, which determines the correlation between the firms' asset returns and the term structure of risk-free interest rates, is set equal to -0.05. This value lies within the range of correlation parameters estimated in empirical studies of structural credit risk models.[15] The credit spreads are set equal to the credit spread means determined by Kiesel et al. (2003).

Second, it is assumed that the portfolio consists of $N = 500$ European call options with counterparty risk on (default) risk-free zero coupon bonds. These are written by 500 different counterparties, but are otherwise identical. The parameters of the risk-free interest rates, the recovery rate, the transition matrix, the asset return correlations as well as the correlation between the asset returns and the interest rates are chosen as above. The simulations are done for the homogeneous initial ratings $\eta_0 \in \{\mathrm{Aa}, \mathrm{Baa}, \mathrm{B}\}$. The expiration date of the options is $T^C = 2$, and the exercise price E is chosen as the $(t = 2)$-forward price of the underlying risk-free zero coupon bond. The risk premia needed for computing the one-year risk-neutral default probabilities are taken from Jarrow et al. (1997, p. 158). All calculations are done on a standard Pentium 4 computer with 1.70 GHz and 512 MB RAM using Visual Basic.

5.2. Results for a portfolio of defaultable zero coupon bonds

In the following, the percentiles $\alpha_{p\%}(L(H))$ of the credit portfolio loss variable $L(H)$ of a portfolio of defaultable zero coupon bonds are computed for $p \in \{95\%, 99\%, 99.9\%, 99.98\%\}$. In each case, this is done with and without an application of the IS techniques. Repeating these computations several times allows to calculate the standard error of the percentile estimators. Based on these standard errors, the ratio of the standard error of the percentile estimator without an application of IS and the respective standard error of the percentile estimator with an application of IS is computed. These ratios allow to evaluate the effectiveness of the IS technique.

As an initial guess for the percentiles y, which we are looking for, the percentiles resulting from a crude pre-Monte Carlo simulation with a very low number of simulation runs (e.g. 10,000) could be used. These initial guesses are needed for computing the IS distributions of the systematic risk factors. The optimal (conditional) parameters $\theta_y\,(Z, X)$ depend on them, too. The exact percentiles are calculated by a simple bisection method. For this, in each iterative step, the arguments in the indicator function of the IS estimators (18) and (43), respectively, are modified until the excess probability estimator equals one minus the confidence level with the desired precision. However, the value of y, on which the parameter $\theta_y(Z, X)$ and the IS means of the systematic risk factors depend, is not altered during the iteration for finding a percentile. When the IS means and IS variances of the market risk factors are chosen as described in Section 4.2.1, an additional pre-simulation is carried out. The resulting guesses y^* for the percentiles are used for calculating the IS means and IS variances of the market risk factors (see (35), (36), and (39)). For this second pre-simulation, the future ratings of the obligors are set equal to their current ratings; consequently, portfolio losses are only due to changes in the market risk factors.

Table 1 shows the standard error ratios when the two-step-estimator (18) is employed. The most important observation is that even for a credit portfolio model with integrated market risk the two-step-IS technique is indeed capable of reducing the standard error of the percentile estimators substantially. Only for the very good credit quality Aa and the extreme asset return correlation $\rho_R = 40\%$, the reduction of the standard error is small. The small sample bias which can occasionally be observed for extremely large asset return correlations and/or very high confidence levels vanishes for a larger number of

[12] For example Barnhill and Maxwell (2002) estimate a short rate volatility of 0.007, whereas Lehrbass (1997) finds $\sigma_r = 0.029$, and Huang and Huang (2003) even work with $\sigma_r = 0.0468$. With regard to the mean reversion parameter and the market price of interest rate risk, Lehrbass finds $\kappa = 1.169$ and absolute values of 0.59, 0.808 and 1.232 for the parameter λ, whereas Huang and Huang choose $\kappa = 0.226$ and an absolute value of 0.248 for λ.

[13] See Moody's Investors Service (1996). However, using Moody's estimate of the mean ignores the fact that the rating agency defines the recovery rate as a percentage of par and not as a percentage of a risk-free but otherwise identical zero coupon bond.

[14] See Moody's Investors Service (2002, p. 31). The probabilities are average values of all corporates in the period 1970–2001. The category 'rating withdrawn' is eliminated by distributing its probability mass among all other categories, corresponding to their individual weights.

[15] See Eom et al. (2004, Table 1, p. 505) and Lyden and Saraniti (2000, Table 6, p. 38).

Table 1
Standard error ratios for percentile estimators with the two-step-IS technique

	Aa, $\rho_R = 10\%$				Aa, $\rho_R = 40\%$			
	$\alpha_{99.98\%}(L)$	$\alpha_{99.9\%}(L)$	$\alpha_{99\%}(L)$	$\alpha_{95\%}(L)$	$\alpha_{99.98\%}(L)$	$\alpha_{99.9\%}(L)$	$\alpha_{99\%}(L)$	$\alpha_{95\%}(L)$
MC^{IS}	17.59	15.41	11.68	8.34	17.92	15.84	11.78	8.37
	*0.0249	*0.0265	*0.0294	*0.0335	*0.7034	*1.0172	*0.0659	*0.0370
	[17.53; 17.65]	[15.34; 15.48]	[11.60; 11.76]	[8.25; 8.43]	[16.11; 19.73]	[13.22; 18.46]	[11.61; 11.95]	[8.27; 8.47]
rt	168				157			
MC	17.90	15.31	11.67	8.33	20.17	15.75	11.79	8.38
	*1.0333	*0.4478	*0.1812	*0.1047	*2.9906	*0.5056	*0.1867	*0.1072
	[15.24; 20.56]	[14.16; 16.46]	[11.20; 12.14]	[8.06; 8.60]	[12.47; 27.87]	[14.45; 17.05]	[11.31; 12.27]	[8.10; 8.66]
rt	53				52			
Ratio	41.5	16.9	6.2	3.1	4.3	0.5	2.8	2.9
	Baa, $\rho_R = 10\%$				Baa, $\rho_R = 40\%$			
	$\alpha_{99.98\%}(L)$	$\alpha_{99.9\%}(L)$	$\alpha_{99\%}(L)$	$\alpha_{95\%}(L)$	$\alpha_{99.98\%}(L)$	$\alpha_{99.9\%}(L)$	$\alpha_{99\%}(L)$	$\alpha_{95\%}(L)$
MC^{IS}	19.67	17.04	12.87	9.29	50.25	32.51	15.57	9.84
	*0.0599	*0.0439	*0.0334	*0.0361	*0.2415	*0.1796	*0.2794	*0.0757
	[19.52; 19.82]	[16.93; 17.15]	[12.78; 12.96]	[9.20; 9.38]	[49.63; 50.87]	[32.05; 32.97]	[14.85; 16.29]	[9.65; 10.03]
rt	128				110			
MC	20.03	16.95	12.85	9.27	53.96	31.98	15.54	9.83
	*1.2745	*0.5036	*0.1983	*0.1128	*10.7274	*2.9405	*0.4141	*0.1439
	[16.75; 23.31]	[15.65; 18.25]	[12.34; 13.36]	[8.98; 9.56]	[26.33; 81.59]	[24.41; 39.55]	[14.47; 16.61]	[9.46; 10.20]
rt	53				54			
Ratio	21.3	11.5	6.0	3.1	44.4	16.4	1.5	1.9
	B, $\rho_R = 10\%$				B, $\rho_R = 40\%$			
	$\alpha_{99.98\%}(L)$	$\alpha_{99.9\%}(L)$	$\alpha_{99\%}(L)$	$\alpha_{95\%}(L)$	$\alpha_{99.98\%}(L)$	$\alpha_{99.9\%}(L)$	$\alpha_{99\%}(L)$	$\alpha_{95\%}(L)$
MC^{IS}	70.33	60.27	44.64	32.66	151.10	133.51	93.51	55.97
	*0.1115	*0.1084	*0.1065	*0.1030	*0.1584	*0.2416	*0.3318	*0.3315
	[70.04; 70.62]	[59.99; 60.55]	[44.37; 44.91]	[32.39; 32.93]	[150.69; 151.51]	[132.89; 134.13]	[92.66; 94.36]	[55.12; 56.82]
rt	87				73			
MC	71.99	60.00	44.61	32.67	152.30	132.52	93.37	55.92
	*4.9163	*1.8952	*0.6798	*0.3345	*6.5357	*4.0189	*2.0336	*1.0868
	[59.33; 84.65]	[55.12; 64.88]	[42.86; 46.36]	[31.81; 33.53]	[135.47; 169.13]	[122.17; 142.87]	[88.13; 98.61]	[53.12; 58.72]
rt	54				53			
Ratio	44.1	17.5	6.9	3.3	41.3	16.6	6.1	3.3

Notes: MC^{IS}: Monte Carlo simulation of the loss variable $L(H)$ for a portfolio of defaultable zero coupon bonds with the two-step-IS technique; MC: Monte Carlo simulation without IS; *: standard error of the percentile estimators; [ ]: 99%-confidence interval of the percentile estimators; rt: run time in seconds (for MC^{IS} without the one-time solution of the optimization problem (17) for finding the IS means); ratio: standard error ratios, defined as the standard error of the percentile estimator without an application of IS divided by the standard error of the respective estimator with IS. The computations of the standard errors are based on 350 repetitions of the simulations (with and without IS), whereby each simulation consists of 10,000 simulation runs. All exhibited percentiles are mean values of these 350 repetitions. *Notation*: ρ_R: asset return correlation. *Parameters*: $N = 500$, $F = 1$, $T = 3$, $H = 1$, $\rho_{X_r,R} = -0.05$, $\delta = 0.538$, $\kappa = 0.4$, $\phi = 0.06$, $\sigma_r = 0.01$, $\lambda = 0.5$, $r(0) = 0.06$.

simulation runs (e.g. 100,000 instead of 10,000). For this increased number of simulation runs, the percentile estimates resulting from the Monte Carlo simulations without IS come up to those resulting from an application of IS (without table).[16] As expected, the reduction of the standard error caused by IS is generally larger, the higher the confidence level of the percentile estimator is. However, no clear dependence of the standard error reduction on the credit quality or the asset return correlation can be observed.[17]

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16 In contrast, changing the number of repetitions (e.g. from 350 to 1000) has hardly any impact (without table).

17 To exclude that this missing dependence of the standard error reduction on the credit quality or the asset return correlation is due to differently exact percentile estimates resulting from the pre-simulations, the exact percentile values are employed for computing the IS means, the optimal (conditional) thetas, and as initial guesses for the iterations. The sensitivity of the standard error reduction effect on these initial guesses was also tested: the same computations, as shown in Table 1, were repeated using the exact percentiles times a factor 0.75 as initial guesses. Of course, the IS means for the systematic risk factors changed, but nevertheless, in general, the reduction effect was still substantial (without table). These observations are in line with the results of GL (2005) for the default mode model. However, using too large percentile estimates as initial guesses (e.g. the exact percentiles times a factor 1.25) can cause an instability and a missing convergence of the two-step-IS approach. GL (2005, p. 1650) do not mention this problem. They only note that one can use the same samples of the loss variable $L(H)$ to estimate excess probabilities $P(L(H) > x)$ at values of x larger than y, but do not comment on the reverse case $x < y$.

Table 2
Relative importance of the IS technique for the two systematic risk factors

Confidence level (%)	Aa		Baa		B	
	$\rho_R = 10\%$	$\rho_R = 40\%$	$\rho_R = 10\%$	$\rho_R = 40\%$	$\rho_R = 10\%$	$\rho_R = 40\%$
No IS for Z						
99.98	39.5	1.5	4.3	0.9	0.9	1.6
99.9	16.9	0.4	3.3	1.1	1.2	0.9
99	6.4	1.2	3.5	1.1	1.1	1.0
95	3.0	2.6	2.6	1.1	1.2	1.0
No IS for X_r						
99.98	1.0	4.5	1.5	34.8	21.0	30.6
99.9	0.9	0.9	0.8	12.4	9.2	14.0
99	1.0	1.0	0.9	1.0	3.8	5.6
95	1.0	1.0	1.1	0.6	2.4	3.0

Notes: Standard error ratios, defined as the standard error of the percentile estimator of the loss variable $L(H)$ for a portfolio of defaultable zero coupon bonds without an application of IS divided by the standard error of the respective estimator with the two-step-IS technique, are shown. In the first case, the IS mean of the systematic credit risk factor Z is set equal to zero and only the optimal IS mean for the interest rate factor X_r is employed. In the second case, the IS mean of X_r is set equal to zero. The computations of the standard errors are based on 350 repetitions of the simulations (with and without IS), whereby each simulation consists of 10,000 simulation runs. *Notation*: ρ_R: asset return correlation. *Parameters*: $N = 500,\ F = 1,\ T = 3,\ H = 1,\ \rho_{X_r,R} = -0.05,\ \delta = 0.538,\ \kappa = 0.4,\ \phi = 0.06,\ \sigma_r = 0.01,\ \lambda = 0.5,\ r(0) = 0.06$.

Under mild smoothness requirements on the distribution function of the credit portfolio loss $L(H)$ in the neighborhood of $\alpha_{p\%}(L(H))$, the sample $p\%$-percentile $\hat{\alpha}_{p\%}(L(H))$ is asymptotically normally distributed with mean $\alpha_{p\%}(L(H))$ and variance $p \cdot (1-p)/D \cdot (f(\alpha_{p\%}(L(H))))^2$ (see Serfling (1980, p. 77)). $f(\cdot)$ denotes the probability density function of the credit portfolio loss $L(H)$, which is assumed to be strictly positive at a neighborhood of the sample percentile $\hat{\alpha}_{p\%}(L(H))$. The nominator $p \cdot (1-p)$ is the variance of the indicator function $1_{\{L(H)>\alpha_{p\%}(L(H))\}}$. Based on the asymptotic normality of $\hat{\alpha}_{p\%}(L(H))$, a large sample $(1-\beta)$-confidence interval for the $p\%$-percentile can be given (see Glasserman (2004, p. 490)):

$$\hat{\alpha}_{p\%}(L(H)) \pm z_{\beta/2} \frac{\sqrt{p \cdot (1-p)}}{\sqrt{D} \cdot f(\alpha_{p\%}(L(H)))}, \tag{48}$$

where $z_{\beta/2} = \Phi^{-1}(1-\beta/2)$. If the portfolio density function $f(\cdot)$ is continuous at $\hat{\alpha}_{p\%}(L(H))$, the confidence interval given by the boundaries in (48) remains asymptotically valid with $f(\alpha_{p\%}(L(H)))$ replaced by $f(\hat{\alpha}_{p\%}(L(H)))$. However, for applying the above confidence interval, the density function at $\hat{\alpha}_{p\%}(L(H))$ has to be estimated. To avoid this, the sample standard error of $\hat{\alpha}_{p\%}(L(H))$ that results from several repetitions of the Monte Carlo simulation is used in (48). These confidence intervals with $\beta = 0.01$ can be seen in Table 1. (48) also shows how to interpret the standard error ratios of the percentile estimators. If the standard error of a percentile estimator without IS is 10 times larger than the standard error with IS, then we need 10^2 times more simulation runs D without IS than we would need with IS to achieve the same precision. Hereby, precision is measured by the standard deviation of the percentile estimator. Table 1 also shows that the Monte Carlo simulation combined with the two-step-IS technique takes more time than the naïve Monte Carlo simulation. The reason is that, for each simulation run, the optimization problem (14) has to be solved for finding the optimal parameter θ.[18] In contrast, the optimization problem (17) for finding the IS means for the systematic risk factors Z and X_r has only to be solved once. Another reason why the IS approach takes more time is that, due to the different IS means of the systematic risk factors for different confidence levels, the simulations for the estimation of the different percentiles have to be done separately for each confidence level.[19]

Table 2 shows the relative importance of IS for the systematic credit risk factor Z and the interest rate factor X_r: for high quality portfolios with a low stochastic dependence between the credit quality changes of the obligors, IS for the interest rate factor X_r is more important, whereas for low credit qualities and/or high asset return correlations, IS for the systematic credit risk factor Z is essential.

It is also tested how effective it is to use only the IS transition probabilities (10) while setting the IS means of the systematic risk factors equal to zero. Doing this, a reduction of the standard errors of the percentile estimators can only be

[18] The increase in run time is larger, the larger the extent of inhomogeneity in the portfolio is. This disadvantage could be avoided if only a one-step-IS procedure is used (see the literature review in Section 2). However, in the end, the trade-off between the variance reduction effect and the run time increase has to be compared for one-step- and two-step-IS algorithms.

[19] Alternatively, to save computation time, one set of IS means could be used for the estimation of all percentiles corresponding to 'higher' confidence levels. GL (2005, p. 1650) report that the variance reduction effect is relatively insensitive with respect to the choice of the initial percentile guess on which the IS means (and the conditional optimal theta values) depend.

Table 3
Standard error ratios for expected shortfall estimators with the two-step-IS technique

Confidence level (%)	Aa		Baa		B	
	$\rho_R = 10\%$	$\rho_R = 40\%$	$\rho_R = 10\%$	$\rho_R = 40\%$	$\rho_R = 10\%$	$\rho_R = 40\%$
99.98	217.2	1.1	62.9	100.3	207.5	276.7
99.9	53.6	0.3	18.5	38.2	44.2	49.6
99	11.4	0.8	9.3	4.6	11.2	10.5
95	5.5	2.1	5.1	4.1	5.3	5.1

Notes: Standard error ratios, defined as the standard error of the estimator for the expected shortfall $E^P[L(H)|L(H) > \alpha_p\%\ (L(H))]$ for a portfolio of defaultable zero coupon bonds without an application of IS divided by the standard error of the respective estimator with the two-step-IS technique, are shown. The computations of the standard errors are based on 350 repetitions of the simulations (with and without IS), whereby each simulation consists of 10,000 simulation runs. *Notation*: ρ_R: asset return correlation. *Parameters*: $N = 500,\ F = 1,\ T = 3,\ H = 1,\ \rho_{X_r,R} = -0.05,\ \delta = 0.538,\ \kappa = 0.4,\ \phi = 0.06,\ \sigma_r = 0.01,\ \lambda = 0.5,\ r(0) = 0.06$.

observed when the factor weights ρ_R and $\rho_{X_r,R}$ and the interest rate volatility σ_r are small (without table). This finding is in line with the theoretical results of GL (2005) who show, on the basis of a one-factor model applied to a homogeneous portfolio, that for larger asset return correlations, IS for the systematic risk factors is more important than IS for the conditional transition probabilities. In contrast to GL's standard default mode model, which they employ for their numerical experiments, in the integrated market and credit portfolio model, which is used here, correlations of the instrument values are also caused by a common dependence on the risk-free discount factor whenever the interest rate volatility σ_r is positive.

Next, the effectiveness of the two-step-IS technique for estimating the expected shortfall, which is – in contrast to the VaR – a coherent risk measure, is tested. It can be shown that the same IS means for the systematic risk factors and the same twisting parameters θ that are optimal for estimating the VaR are also optimal (in the sense of Section 4.1) for estimating the expected shortfall. Table 3 shows the standard error ratios for the estimators of $E^P[L(H)|L(H) > y]$ with $y = \alpha_{p\%}(L(H))$ ($p \in \{95\%, 99\%, 99.9\%, 99.98\%\}$). In general, the standard error reduction effect is substantially strengthened for this risk measure. Without an application of IS, even the mean of the expected shortfall estimator with $p = 99.98\%$ over all 350 repetitions is so poor that it is not of any use (without table). However, this is not surprising because with 10,000 simulation runs, on average, only two realizations of the credit portfolio value are larger than the threshold $y = \alpha_{99.98\%}(L(H))$. Thus, on average, only two out of 10,000 simulation runs are relevant for the computation of the expected shortfall when IS is not used. For an increased number of simulation runs (e.g. 100,000 instead of 10,000), the expected shortfall estimates resulting from the Monte Carlo simulations without IS come up to those resulting from an application of IS (without table).[20]

Next, the influence of the model's parameterization and the homogeneity assumption on the standard error reduction effect is tested. More specifically, the influence of changes in the amount of interest rate risk, the correlation between the asset returns and the risk-free interest rates, the number of obligors, and the degree of homogeneity in the portfolio is analyzed. The result is that some standard error ratios are larger than in the base case setting, others are lower, but no systematic difference can be observed (compare with Table 1). Even for the very good credit quality Aa and the extreme asset return correlation $\rho_R = 40\%$, a substantial variance reduction effect, which is larger than that one in the base case setting, can be observed for some parameterizations (without table). This indicates that, to some extent, the effectiveness of the two-step-IS technique depends on the parameters of the model (see Table 4.).

The effectiveness of the two-step-IS technique is also tested when IS for the conditional transition probabilities is assumed to be done only while determining the IS means for the systematic risk factors, but not while the actual simulations later. Surprisingly, in many cases, a similar or only slightly smaller reduction of the standard error of the percentile estimators than with the full two-step-IS technique can be observed (without table). Checking the optimal theta values, which result from setting the systematic risk factors equal to their IS means, this result is not too surprising any more. Frequently, these optimal theta values are zero or slightly above zero. This explains why the effect from setting theta equal to zero during the simulations is not too large. This observation raises doubt whether a two-step-IS technique is really necessary or whether simpler one-step-IS techniques[21] might not be as effective as the two-step-IS approach.

Finally, the combined three-step-IS technique described in Section 4.2 is implemented. The IS mean of the systematic credit risk factor Z is calculated according to (47), whereas the computation of the IS mean and IS variance of the interest rate factor X_r is based on (35), (36), and (39). As the second derivative $\partial^2 L^{\text{wtr}}(X_r, H)/(\partial X_r)^2|_{X_r = E^P[X_r]=0}$ is very small, the parameter λ_1 is nearly zero. This implies that the IS variance (36) remains, compared to the original probability measure, nearly unchanged one. Table 5 shows that for almost all considered credit qualities, asset return correlations, and confidence levels, the combined three-step-IS technique yields worse standard error reductions than the two-step-IS approach.

[20] However, in this case, the standard error ratios are also reduced.

[21] See the literature review in Section 2.

Table 4
Robustness checks for the standard error ratios for percentile estimators with the two-step-IS technique

Confidence level (%)	Aa			Baa			B		
Interest rate volatility	$\sigma_r = 0$	$\sigma_r = 0.005$	$\sigma_r = 0.02$	$\sigma_r = 0$	$\sigma_r = 0.005$	$\sigma_r = 0.02$	$\sigma_r = 0$	$\sigma_r = 0.005$	$\sigma_r = 0.02$
99.98	47.2	37.3	40.0	48.9	41.6	35.1	40.3	40.3	42.3
99.9	16.4	15.9	16.9	17.6	14.3	14.9	17.3	17.2	16.5
99	7.2	5.6	6.1	6.4	5.6	5.8	6.5	6.6	5.9
95	2.7	3.1	3.0	3.4	3.1	3.1	3.4	3.3	2.9

Correlation between asset returns and interest rates	Aa $\rho_{X_r,R} = -0.25$	Aa $\rho_{X_r,R} = 0.25$	Baa $\rho_{X_r,R} = -0.25$	Baa $\rho_{X_r,R} = 0.25$	B $\rho_{X_r,R} = -0.25$	B $\rho_{X_r,R} = 0.25$
99.98	41.1	40.9	40.9	35.4	43.2	42.9
99.9	16.5	15.2	17.6	16.6	16.7	17.3
99	6.0	6.6	5.8	6.1	5.7	6.7
95	3.2	3.3	3.0	3.4	3.0	3.2

Number of obligors N=50	Aa	Baa	B
99.98	40.2	35.2	38.4
99.9	18.3	13.4	17.2
99	6.5	6.2	7.2
95	2.9	3.2	3.3

Inhomogeneous portfolio composition	
99.98	39.8
99.9	15.9
99	5.6
95	2.9

Notes: Standard error ratios, defined as the standard error of the percentile estimator of the loss variable $L(H)$ for a portfolio of defaultable zero coupon bonds without an application of IS divided by the standard error of the respective estimator with the two-step-IS technique, are shown. The computations of the standard errors are based on 350 repetitions of the simulations (with and without IS), whereby each simulation consists of 10,000 simulation runs. The inhomogeneous portfolio contains the following numbers n_j of bonds, whose issuers have the initial rating j and whose face values are F_j: $(n_{\mathrm{Aaa}}, F_{\mathrm{Aaa}}) = (15,1)$,$(n_{\mathrm{Aa}}, F_{\mathrm{Aa}}) = (25,1)$,$(n_{\mathrm{A}}, F_{\mathrm{A}}) = (65,5)$,$(n_{\mathrm{Baa}}, F_{\mathrm{Baa}}) = (155,10)$,$(n_{\mathrm{Ba}}, F_{\mathrm{Ba}}) = (165,10)$,$(n_{\mathrm{B}}, F_{\mathrm{B}}) = (55,5)$,$(n_{\mathrm{Caa}}, F_{\mathrm{Caa}}) = (20,1)$. *Parameters (as far as not otherwise indicated)*: $N = 500,\ F = 1,\ T = 3,\ H = 1,\ \rho_R = 0.1,\ \rho_{X_r,R} = -0.05,\ \delta = 0.538,\ \kappa = 0.4,\ \phi = 0.06,\ \sigma_r = 0.01,\ \lambda = 0.5,\ r(0) = 0.06.$

Only for an initial rating Aa and an asset return correlation of 10%, which is the scenario in which interest rate risk has its largest importance (see Table 2), the results are comparable with the two-step-IS technique. One reason for this worse performance of the three-step-IS technique is that the approach for computing the optimal twisting parameter θ when calculating the IS mean of Z is not identical with the approach for computing this parameter during each of the simulation runs. In the former case, the interest rate factor is assumed to be non-stochastic and set equal to its IS mean. In contrast, in the latter case, θ is chosen as a function of the realizations of the systematic risk factors Z and X_r, which can both vary in a free manner according to their probability distribution. Hence, when determining the optimal IS mean of Z only a sub-optimal parameter θ (and not the optimal value characterized by (14)) is employed. From this, the sub-optimality of the computed IS mean of Z follows. Furthermore, the IS mean of the interest rate factor X_r chosen according to (35) hardly depends on the initial rating of the obligors. The consequence is that no reduction of the IS mean of X_r takes place when more importance has to be put on the credit risk factor Z. For example, for the initial rating B, an asset return correlation of $\rho_R = 10\%$, and a confidence level of 99.98%, the optimal two-step-IS means would be $(\mu_Z^{2\ \mathrm{step}}, \mu_{X_r}^{2\ \mathrm{step}}) = (-3.5, 1.1)$. However, the IS means employed for the three-step-IS technique are $(\mu_Z^{3\ \mathrm{step}}, \mu_{X_r}^{3\ \mathrm{step}}) = (-2.6, 3.5)$. The consequence is that a reduction of the standard error of the percentile estimators can hardly be observed any more for this scenario. In contrast, for the initial rating Aa, $\rho_R = 10\%$, and a confidence level of 99.98%, the difference between the optimal two-step-IS means $(\mu_Z^{2\ \mathrm{step}}, \mu_{X_r}^{2\ \mathrm{step}}) = (0, 3.6)$ and those IS means $(\mu_Z^{3\ \mathrm{step}}, \mu_{X_r}^{3\ \mathrm{step}}) = (-0.6, 3.5)$ employed for the three-step-IS technique is not too large. As a consequence, for this scenario, the standard error reduction effect produced by the three-step-IS technique is still substantial as Table 5 shows.

5.3. Results for a portfolio of European call options with counterparty risk

Next, the two-step-IS technique is applied to an interest rate option portfolio. However, as the main reason for losses in an option portfolio is a negative development of the market risk factors underlying the options, the risk measures are hardly sensitive to changes in the credit quality of the issuers or the asset return correlation. These findings are in line with those of Duffie and Pan (2001, Table 5, Fig. 6a). For the two-step-IS technique, the consequence is that the optimization

Table 5
Standard error ratios for percentile estimators with the combined three-step-IS technique

Confidence level (%)	Aa		Baa		B	
	$\rho_R = 10\%$	$\rho_R = 40\%$	$\rho_R = 10\%$	$\rho_R = 40\%$	$\rho_R = 10\%$	$\rho_R = 40\%$
99.98	36.6	3.2	13.1	1.5	2.6	1.8
99.9	17.0	2.3	9.0	0.9	1.3	1.0
99	6.2	3.6	5.7	0.8	1.4	0.7
95	3.2	2.4	3.1	2.4	1.6	1.0

Notes: Standard error ratios, defined as the standard error of the percentile estimator of the loss variable $L(H)$ for a portfolio of defaultable zero coupon bonds without an application of IS divided by the standard error of the respective estimator with the combined three-step-IS technique, are shown. The computations of the standard errors are based on 350 repetitions of the simulations (with and without IS), whereby each simulation consists of 10,000 simulation runs. The IS means of the systematic credit risk factor Z are calculated according to (47), whereas the computations of the IS means and variances of the interest rate factor X_r are based on (35), (36), and (39). *Notation*: ρ_R: asset return correlation. *Parameters*: $N = 500,\ F = 1,\ T = 3,\ H = 1,\ \rho_{X_r,R} = -0.05,\ \delta = 0.538,\ \kappa = 0.4,\ \phi = 0.06,\ \sigma_r = 0.01,\ \lambda = 0.5,\ r(0) = 0.06$.

Table 6
Admissible intervals for the initial guess y and optimal θ values

X_r	Confidence level 99.98%, exact percentile: 1.725242		X_r	Confidence level 95%, exact percentile: 1.378290	
	Interval	θ^{opt}		Interval	θ^{opt}
3.1	(1.700319; 1.721899)	∞	1.0	(1.004624; 1.347275)	∞
3.2	(1.707588; 1.725813)	>15,000	1.1	(1.075470; 1.385424)	>15,000
3.3	(1.713848; 1.729184)	>15,000	1.2	(1.141300; 1.420872)	>15,000
3.4	(1.719217; 1.732075)	>15,000	1.5	(1.310348; 1.511902)	>15,000
3.5	(1.723804; 1.734545)	>15,000	1.6	(1.357837; 1.537475)	>15,000
3.6	(1.727708; 1.736648)	0	*1.7*	(1.401236; 1.560845)	0
3.7	(1.731017; 1.738430)	0	1.8	(1.440758; 1.582127)	0

Notes: Table 6 shows for a homogeneous portfolio of European call options with counterparty risk on (default) risk-free zero coupon bonds the admissible intervals (according to (26)) for the initial percentile guess y. The issuers of the options have an initial rating of Aa and exhibit an asset return correlation of $\rho_R = 10\%$. Furthermore, the solution θ^{opt} of the optimization problem (14) when y is set equal to the exact percentile is shown for various scenarios of the systematic risk factors. The systematic credit risk factor Z is set equal to zero, which is its IS mean in all cases. The values of the interest rate factor X_r are chosen around its IS mean (cursive in the above table). For computing the IS means as well as for the above calculations, the maximum value of θ is set equal to 15,000. *Parameters*: $N = 500,\ F = 1,\ T = 3,\ T^C = 2,\ H = 1,\ E = 0.934498,\ \rho_{X_r,R} = -0.05,\ \delta = 0.538,\ \kappa = 0.4,\ \phi = 0.06,\ \sigma_r = 0.01,\ \lambda = 0.5,\ r(0) = 0.06$.

problem (14) might not be well defined. This optimization problem has a solution $\theta \in (0,\infty)$ only for initial guesses y of the percentile that are larger than the conditional expected loss and smaller than the conditional maximum loss (see (26)). However, as Table 6 shows, this interval can be rather small for an option portfolio, especially for higher confidence levels. And even if the initial guess is in the interval (26), the optimal parameter θ fulfilling (14) can be very large, which can cause overflow problems during the simulations. For example, setting the systematic risk factors (Z, X_r) equal to $(0, 3.5)$ for an initial rating Aa and a confidence level of 99.98% yields non-zero values for θ only for $y > 1.723804$. However, for the slightly increased initial guess $y = 1.723805$, the optimal value for θ is already 14,580. The reason for these large θ values is the relatively small influence of rating transitions on the loss of the option when the interest rate factor X_r is around its IS mean.

Despite these difficulties, the performance of the two-step-IS technique is also tested for an interest rate option portfolio.[22] For this, a maximum value for the parameter θ is defined: whenever the true value is larger than 100, $\theta = 100$ is set. As Table 7 shows, the two-step-IS technique still yields a substantial reduction of the standard error of the percentile estimators.[23]

[22] The three-step-IS technique is not applied to the portfolio of interest options because of the disappointing results reported in the previous section. Furthermore, due to the bad quality of the delta–gamma approximation over a long risk horizon of 1 year, it is not possible to compute the IS distribution of the interest rate factor X_r for higher confidence levels.

[23] However, for the option portfolio, more frequently the missing convergence of the iteration, mentioned already before, is observed, even if the exact percentiles are used as initial guesses. In these cases, the optimal parameter $\theta_y(Z^{(d)}, X_r^{(d)})$ is too often too large which implies that the conditional likelihood ratio $\exp(-\theta_y(Z^{(d)}, X_r^{(d)})L(H)^{(d)} + \psi_{L(H)|Z^{(d)},X_r^{(d)}}(\theta_y(Z^{(d)}, X_r^{(d)})))$ is too often almost or identical to zero so that not enough probability mass is produced. This problem occurs when the maximum value for theta is chosen too large (e.g. $\theta^{\max} = 15{,}000$).

Table 7
Standard error ratios for percentile estimators of an interest rate option portfolio with counterparty risk with the two-step-IS technique

	$\alpha_{99.98\%}(L)$	$\alpha_{99.9\%}(L)$	$\alpha_{99\%}(L)$	$\alpha_{95\%}(L)$	$\alpha_{99.98\%}(L)$	$\alpha_{99.9\%}(L)$	$\alpha_{99\%}(L)$	$\alpha_{95\%}(L)$
	Aa, $\rho_R = 10\%$				Aa, $\rho_R = 40\%$			
MC^{IS}	1.7255	1.6996	1.5938	1.3780	1.7254	1.6995	1.5936	1.3776
	*0.0002	*0.0004	*0.0012	*0.0028	*0.0002	*0.0004	*0.0013	*0.0030
MC	1.7266	1.6979	1.5929	1.3775	1.7268	1.6981	1.5935	1.3775
	*0.0072	*0.0069	*0.0082	*0.0089	*0.0073	*0.0071	*0.0079	*0.0094
ratio	36.3	17.2	6.7	3.2	33.5	16.0	5.9	3.2
	Baa, $\rho_R = 10\%$				Baa, $\rho_R = 40\%$			
MC^{IS}	1.7206	1.6948	1.5894	1.3744	1.7206	1.6948	1.5894	1.3743
	*0.0002	*0.0004	*0.0013	*0.0030	*0.0002	*0.0004	*0.0013	*0.0029
MC	1.7216	1.6927	1.5885	1.3738	1.7219	1.6930	1.5887	1.3738
	*0.0073	*0.0073	*0.0081	*0.0092	*0.0073	*0.0076	*0.0081	*0.0090
ratio	35.1	17.3	6.1	3.1	35.4	17.5	6.2	3.1
	B, $\rho_R = 10\%$				B, $\rho_R = 40\%$			
MC^{IS}	1.7046	1.6801	1.5796	1.3740	1.7047	1.6803	1.5802	1.3751
	*0.0002	*0.0004	*0.0013	*0.0029	*0.0002	*0.0004	*0.0013	*0.0028
MC	1.7057	1.6787	1.5792	1.3738	1.7058	1.6785	1.5794	1.3751
	*0.0067	*0.0070	*0.0074	*0.0086	*0.0068	*0.0072	*0.0076	*0.0087
ratio	32.6	16.0	5.7	2.9	35.1	17.9	6.1	3.2

Notes: MC^{IS}: Monte Carlo simulation of the loss variable $L(H)$ for a portfolio of European call options with counterparty risk on (default) risk-free zero coupon bonds with the two-step-IS technique; MC: Monte Carlo simulation without IS; *: standard error of the percentile estimators; ratio: standard error ratios, defined as the standard error of the percentile estimator without an application of IS divided by the standard error of the respective estimator with IS. For the computation of the IS means as well as for the simulation, the maximum value for θ is set equal to 100. The computations of the standard errors are based on 350 repetitions of the simulations (with and without IS), whereby each simulation consists of 10,000 simulation runs. All exhibited percentiles are mean values of these 350 repetitions. *Notation*: ρ_R: asset return correlation. *Parameters*: $N = 500, F = 1, T = 3, T^C = 2, H = 1, E = 0.934498, \rho_{X_r,R} = -0.05, \delta = 0.538, \kappa = 0.4, \phi = 0.06, \sigma_r = 0.01, \lambda = 0.5, r(0) = 0.06$.

6. Conclusions

In this paper, the benefit which results from applying IS techniques to an integrated market and credit portfolio model is analyzed. It is shown in detail how the two-step-IS technique of GL (2005) can be adjusted to a general integrated market and credit portfolio model. Furthermore, it is discussed how an IS approach originally developed for pure market risk portfolio models can be combined with the two-step-IS approach to build up a potentially even more effective three-step-IS technique. The effectiveness of the various approaches when estimating large percentiles of the credit portfolio loss variable is tested by means of numerical experiments.

The main result is that the two-step-IS technique of GL (2005) originally developed for pure default mode credit portfolio models can basically also be applied to integrated market and credit portfolio models. As expected, this technique can substantially reduce the dispersion of the percentile estimators, even in the context of an integrated market and credit portfolio model. The combination of the two-step-IS technique of GL (2005) with an IS technique originally developed for pure market risk portfolio models is less effective than the simpler two-step-IS method when applied to a bond portfolio (at least for the chosen implementation of the method in this paper). This result is rather unexpected. The problem with this combined method seems to be the merging step. For an interest rate option portfolio, the IS technique originally developed for pure market risk portfolio models is not applicable at all due to the bad quality of the delta–gamma approximation over a long risk horizon of 1 year.

However, the previous analysis also reveals that the two-step-IS technique of GL (2005) adjusted to an integrated market and credit portfolio model is not necessarily the ideal choice. Due to the market risk dependency of the losses, numerical problems can arise during the solution of the optimization problem (14). This problem could be observed for an option portfolio with counterparty risk and made an ad hoc adjustment of the IS technique necessary. These results indicate that it might be problematic to simply transfer computational techniques originally developed either for pure credit risk portfolio models or for pure market risk portfolio models to the extended class of integrated market and credit portfolio models. Furthermore, the solution of the optimization problem (14) in each simulation run increases the computational burden of this IS technique compared to simpler one-step-IS methods, in particular in the case of very inhomogeneous credit portfolios. Thus, in future research, the performance of these one-step-IS approaches with respect to their ability to reduce the standard error of the percentile estimators as well as the required computational time should be compared with the performance of the two-step-IS technique employed in this paper. Doing this performance comparison, the effect of the number of systematic risk factors on each method's performance should also be analyzed.

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