

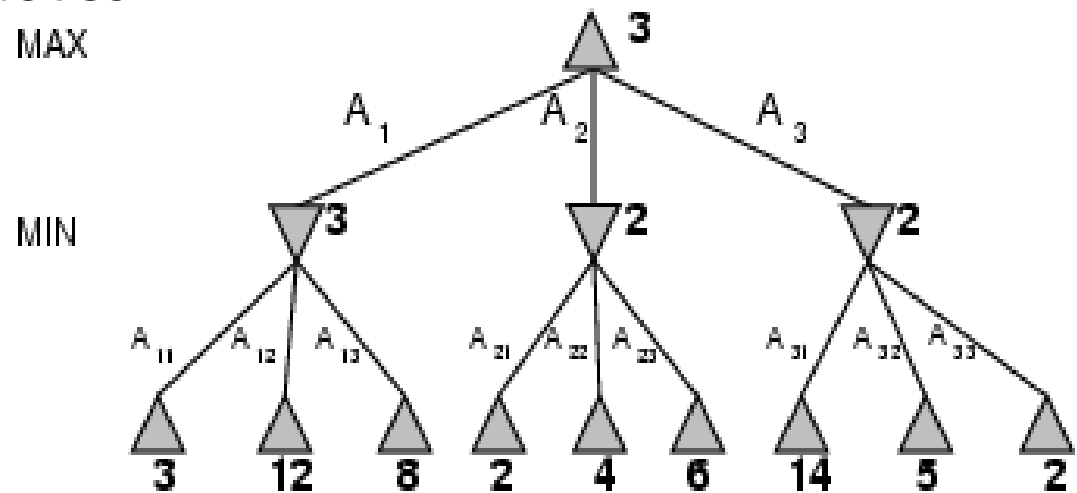
Game Theory: An Introduction

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Is this game theory?

- We've learnt minimax (adversarial search)
 - Perfect play for deterministic games
 - Idea: Choose move to a position with highest minimax value.
 - Goal: Find optimal moves



What is game theory?

- We focus on games where:
 - There are at least two **rational** players
 - Each player has more than one **choices (moves, strategies)**
 - The outcome depends on the strategies chosen by all players; there is **strategic interaction**
- Example: Six people go to a restaurant.
 - Each person pays his/her own meal – **a simple decision problem**
 - Before the meal, every person agrees to split the bill evenly among them – **a game**

What is game theory?

- **Game theory** is a formal way to analyze strategic interaction among a group of rational players (or agents) who behave strategically
- Game theory has applications
 - Economics
 - Politics
 - Biology
 - Computer Science
 - Etc...

Nobel Prizes

- 1994, John Harsanyi, John Forbes Nash and Reinhard Selten, WON NOBEL PRIZE IN ECONOMIC SCIENCES, “for their **pioneering analysis of equilibria** in the theory of non-cooperative games”.
- 2005, Robert Aumann, Thomas Schelling, WON NOBEL PRIZE IN ECONOMIC SCIENCES, “for having enhanced our understanding of **conflict and cooperation** through game theory analysis”.
- 2007, Leonid Hurwicz, Eric Maskin and Roger Myerson, WON NOBEL PRIZE IN ECONOMIC SCIENCES, “for having laid the foundations of **mechanism design** theory”.

Game Theory vs. Combinatorial Game Theory

- They **DIFFER** in nature!
 - In this lecture, we focus on game theory.
 - Combinatorial Game Theory only studies **two-player games** which have a position in which the player take turns changing in defined ways or moves to achieve a defined winning condition.
 - It deals with games with **complete** information, **deterministic** and **where players move** in turn.
- Some combinatorial game theory examples
 - Blue-Red Hackenbush
 - Blue-Red-Green Hackenbush
 - Nim
 - Sprague Grundy Theorem
- These topics are not going to be covered in my lecture in that they focus on different aspects of game theory.

One funny example of combinatorial game – SOS Game

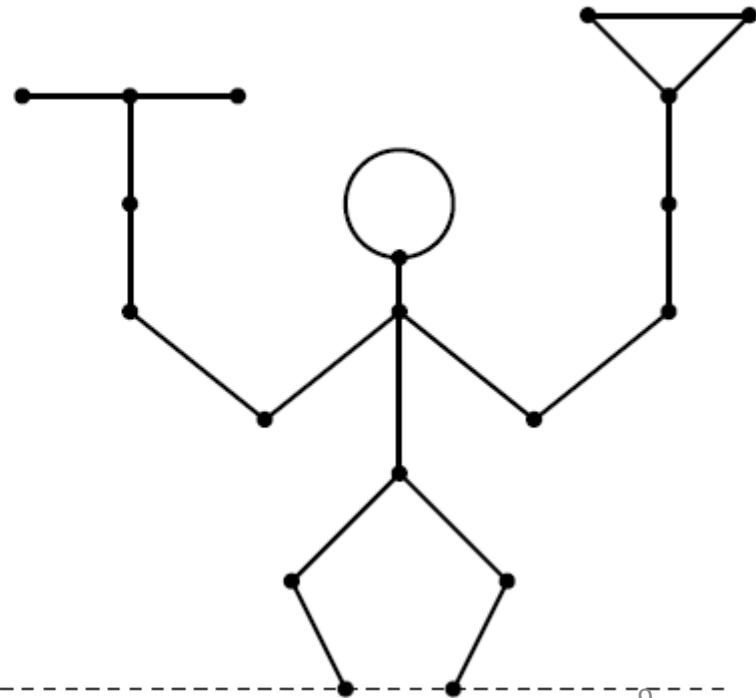
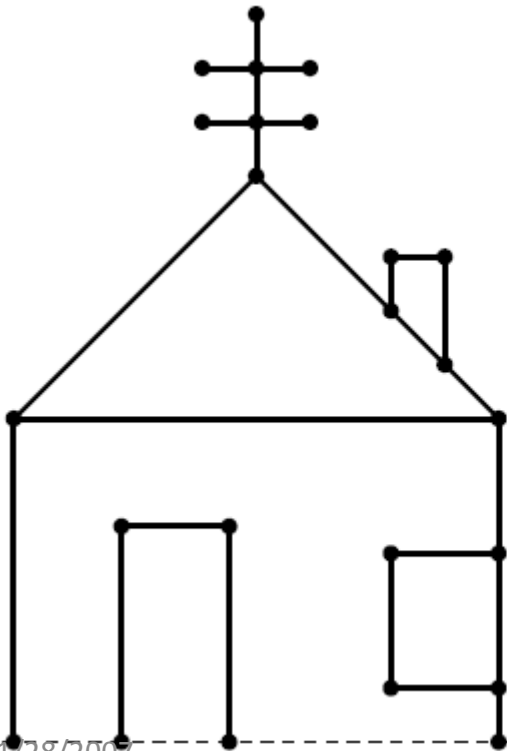
- The SOS Game (From the 28th Annual USA Mathematical Olympiad, 1999)
 - The board consists of a row of n squares, initially empty.
 - Players take turns selecting an empty square and writing either an S or an O in it.
 - The player who first succeeds in completing SOS in consecutive squares wins the game.
 - If the whole board gets filled up without an SOS appearing consecutively anywhere, the game is a draw.
- Given the value of n , discuss the outcome of the game if two players both play in an optimal way.
 - (a) Suppose $n = 4$, the first player puts an S in the first square.
 - (b) Suppose $n = 7$.
 - (c) Suppose $n = 2000$.
 - (d) Suppose $n = 14$.

SOS Game as adversarial search

- 2D SOS Game
 - <http://crystal.uta.edu/~athitsos/courses/cse4308/assignments/assignment4/index.html>
 - Maybe we can use it as next year's programming assignment 1. 😊

Another funnier example – Green Hackenbush

- Hacking away edges from a rooted graph and removing those pieces of the graph that are no longer connected to the ground.



Classic Example: Prisoners' Dilemma

- Two suspects **held in separate cells** are charged with a major crime. However, there is not enough evidence.
- Both suspects are told the following policy:
 - If neither confesses then both will be convicted of a minor offense and sentenced to one month in jail.
 - If both confess then both will be sentenced to jail for six months.
 - If one confesses but the other does not, then the confessor will be released but the other will be sentenced to jail for nine months.

		Prisoner 2	
		Mum	Confess
Prisoner 1	Mum	-1 , -1	-9 , 0
	Confess	0 , -9	-6 , -6

Example: The battle of the sexes

- At the **separate** workplaces, Chris and Pat must choose to attend either an opera or a prize fight in the evening.
- Both Chris and Pat know the following:
 - Both would like to spend the evening together.
 - But Chris prefers the opera.
 - Pat prefers the prize fight.

		Pat	
		Opera	Prize Fight
Chris	Opera	2 , 1	0 , 0
	Prize Fight	0 , 0	1 , 2

Example: Matching pennies

- Each of the two players has a penny.
- Two players must **simultaneously** choose whether to show the Head or the Tail.
- Both players know the following rules:
 - If two pennies match (both heads or both tails) then player 2 wins player 1's penny.
 - Otherwise, player 1 wins player 2's penny.

		Player 2	
		Head	Tail
Player 1	Head	-1 , 1	1 , -1
	Tail	1 , -1	-1 , 1

Static (or simultaneous-move) games of complete information

A static (or simultaneous-move) game consists of:

- A set of players (at least two players)
 - {Player 1, Player 2, ... Player n }
- For each player, a set of strategies/actions
 - $S_1 S_2 \dots S_n$
- Payoffs received by each player for the combinations of the strategies, or for each player, preferences over the combinations of the strategies
 - $u_i(s_1, s_2, \dots, s_n)$, for all $s_1 \in S_1, s_2 \in S_2, \dots, s_n \in S_n$.

Static (or simultaneous-move) games of complete information

- Simultaneous-move
 - Each player chooses his/her strategy without knowledge of others' choices.
- Complete information
 - Each player's strategies and payoff function are common knowledge among all the players.
- Assumptions on the players
 - Rationality
 - Players aim to maximize their payoffs
 - Players are perfect calculators
 - Each player knows that other players are rational

Static (or simultaneous-move) games of complete information

- The players cooperate?
 - No. Only noncooperative games
- The timing
 - Each player i chooses his/her strategy s_i without knowledge of others' choices.
 - Then each player i receives his/her payoff $u_i(s_1, s_2, \dots, s_n)$.
 - The game ends.

Definition: normal-form or strategic-form representation

- The *normal-form* (or *strategic-form*) *representation* of a game G specifies:
 - A finite set of players $\{1, 2, \dots, n\}$,
 - players' strategy spaces $S_1 S_2 \dots S_n$ and
 - their payoff functions $u_1 u_2 \dots u_n$
where $u_i : S_1 \times S_2 \times \dots \times S_n \rightarrow R$.

Normal-form representation: 2-player game

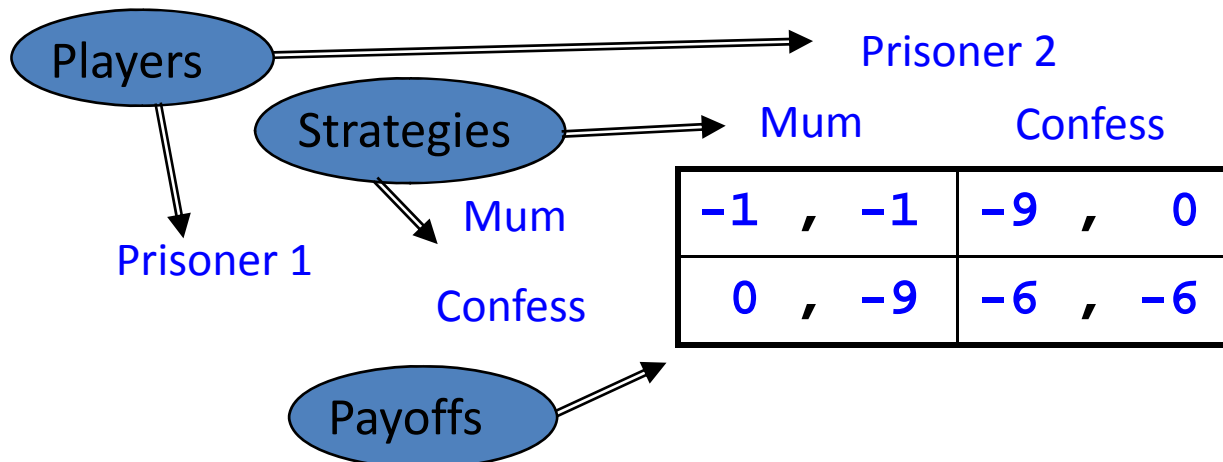
- Bi-matrix representation
 - 2 players: Player 1 and Player 2
 - Each player has a finite number of strategies
- Example:

$$S_1 = \{s_{11}, s_{12}, s_{13}\} \quad S_2 = \{s_{21}, s_{22}\}$$

		Player 2	
		s_{21}	s_{22}
Player 1	s_{11}	$u_1(s_{11}, s_{21}), u_2(s_{11}, s_{21})$	$u_1(s_{11}, s_{22}), u_2(s_{11}, s_{22})$
	s_{12}	$u_1(s_{12}, s_{21}), u_2(s_{12}, s_{21})$	$u_1(s_{12}, s_{22}), u_2(s_{12}, s_{22})$
	s_{13}	$u_1(s_{13}, s_{21}), u_2(s_{13}, s_{21})$	$u_1(s_{13}, s_{22}), u_2(s_{13}, s_{22})$

Classic example: Prisoners' Dilemma: normal-form representation

- Set of players: $\{\text{Prisoner 1, Prisoner 2}\}$
- Sets of strategies: $S_1 = S_2 = \{\underline{\text{Mum}}, \underline{\text{Confess}}\}$
- Payoff functions:
 $u_1(\text{M}, \text{M})=-1, u_1(\text{M}, \text{C})=-9, u_1(\text{C}, \text{M})=0, u_1(\text{C}, \text{C})=-6;$
 $u_2(\text{M}, \text{M})=-1, u_2(\text{M}, \text{C})=0, u_2(\text{C}, \text{M})=-9, u_2(\text{C}, \text{C})=-6$



Example: The battle of the sexes

		Pat	
		Opera	Prize Fight
Chris	Opera	2 , 1	0 , 0
	Prize Fight	0 , 0	1 , 2

- Normal (or strategic) form representation:
 - Set of players: $\{ \text{Chris}, \text{Pat} \}$ ($=\{\text{Player 1}, \text{Player 2}\}$)
 - Sets of strategies: $S_1 = S_2 = \{ \underline{O}$ pera, Prize Fight $\}$
 - Payoff functions:

 $u_1(O, O)=2, u_1(O, F)=0, u_1(F, O)=0, u_1(F, F)=1;$

 $u_2(O, O)=1, u_2(O, F)=0, u_2(F, O)=0, u_2(F, F)=2$

Example: Matching pennies

		Player 2	
		Head	Tail
Player 1	Head	-1 , 1	1 , -1
	Tail	1 , -1	-1 , 1

- Normal (or strategic) form representation:
 - Set of players: {Player 1, Player 2}
 - Sets of strategies: $S_1 = S_2 = \{ \underline{H}ead, \underline{T}ail \}$
 - Payoff functions:
 $u_1(H, H)=-1, u_1(H, T)=1, u_1(T, H)=1, u_1(T, T)=-1;$
 $u_2(H, H)=1, u_2(H, T)=-1, u_2(T, H)=-1, u_2(T, T)=1$

Example: Tourists & Natives

- Only two bars (bar 1, bar 2) in a city
- Can charge price of \$2, \$4, or \$5
- 6000 tourists pick a bar randomly
- 4000 natives select the lowest price bar

- Example 1: Both charge \$2
 - each gets 5,000 customers and \$10,000
- Example 2: Bar 1 charges \$4, Bar 2 charges \$5
 - Bar 1 gets $3000+4000=7,000$ customers and \$28,000
 - Bar 2 gets 3000 customers and \$15,000

Example: Cournot model of duopoly

- A product is produced by only two firms: firm 1 and firm 2. The quantities are denoted by q_1 and q_2 , respectively. Each firm chooses the quantity without knowing the other firm has chosen.
- The market price is $P(Q)=a-Q$, where $Q=q_1+q_2$.
- The cost to firm i of producing quantity q_i is $C_i(q_i)=cq_i$.

The normal-form representation:

- Set of players: $\{\text{Firm 1, Firm 2}\}$
- Sets of strategies: $S_1=[0, +\infty), S_2=[0, +\infty)$
- Payoff functions:
 $u_1(q_1, q_2)=q_1(a-(q_1+q_2)-c), u_2(q_1, q_2)=q_2(a-(q_1+q_2)-c)$

One More Example

- Each of n players selects a number between 0 and 100 simultaneously. Let x_i denote the number selected by player i .
- Let y denote the average of these numbers
- Player i 's payoff = $\text{abs}(x_i - 3y/5)$.
- The normal-form representation:

Solving Prisoners' Dilemma

- Confess always does better whatever the other player chooses
- Dominated strategy
 - There exists another strategy which always does better regardless of other players' choices

		Prisoner 2	
		Mum	Confess
Prisoner 1	Mum	-1 , -1	-9 , 0
	Confess	0 , -9	-6 , -6

Definition: strictly dominated strategy

In the normal-form game $\{S_1, S_2, \dots, S_n, u_1, u_2, \dots, u_n\}$, let $s_i', s_i'' \in S_i$ be feasible strategies for player i .

Strategy s_i' is *strictly dominated* by strategy s_i'' if

$$u_i(s_1, s_2, \dots, s_{i-1}, s_i', s_{i+1}, \dots, s_n) < u_i(s_1, s_2, \dots, s_{i-1}, s_i'', s_{i+1}, \dots, s_n)$$

s_i'' is strictly better than s_i'

for all $s_1 \in S_1, s_2 \in S_2, \dots, s_{i-1} \in S_{i-1}, s_{i+1} \in S_{i+1}, \dots, s_n \in S_n$.

regardless of other players' choices

Prisoner 2

		Prisoner 2	
		Mum	Confess
Prisoner 1	Mum	-1 , -1	-9 , 0
	Confess	0 , -9	-6 , -6

Example

- Two firms, Reynolds and Philip, share some market
- Each firm earns \$60 million from its customers if neither do advertising
- Advertising costs a firm \$20 million
- Advertising captures \$30 million from competitor

		Philip	
		No Ad	Ad
Reynolds	No Ad	60 , 60	30 , 70
	Ad	70 , 30	40 , 40

2-player game with finite strategies

- $S_1 = \{s_{11}, s_{12}, s_{13}\}$ $S_2 = \{s_{21}, s_{22}\}$
- s_{11} is strictly dominated by s_{12} if $u_1(s_{11}, s_{21}) < u_1(s_{12}, s_{21})$ and $u_1(s_{11}, s_{22}) < u_1(s_{12}, s_{22})$.
- s_{21} is strictly dominated by s_{22} if $u_2(s_{1i}, s_{21}) < u_2(s_{1i}, s_{22})$, for $i = 1, 2, 3$

Player 2

		Player 2	
		s_{21}	s_{22}
Player 1	s_{11}	$u_1(s_{11}, s_{21}), u_2(s_{11}, s_{21})$	$u_1(s_{11}, s_{22}), u_2(s_{11}, s_{22})$
	s_{12}	$u_1(s_{12}, s_{21}), u_2(s_{12}, s_{21})$	$u_1(s_{12}, s_{22}), u_2(s_{12}, s_{22})$
	s_{13}	$u_1(s_{13}, s_{21}), u_2(s_{13}, s_{21})$	$u_1(s_{13}, s_{22}), u_2(s_{13}, s_{22})$

Definition: weakly dominated strategy

In the normal-form game $\{S_1, S_2, \dots, S_n, u_1, u_2, \dots, u_n\}$, let $s_i', s_i'' \in S_i$ be feasible strategies for player i . Strategy s_i' is **weakly dominated** by strategy s_i'' if

$$u_i(s_1, s_2, \dots, s_{i-1}, s_i', s_{i+1}, \dots, s_n) \leq (\text{but not always } =) u_i(s_1, s_2, \dots, s_{i-1}, s_i'', s_{i+1}, \dots, s_n)$$

for all $s_1 \in S_1, s_2 \in S_2, \dots, s_{i-1} \in S_{i-1}, s_{i+1} \in S_{i+1}, \dots, s_n \in S_n$.

s_i'' is at least as good as s_i'

regardless of other players' choices

		Player 2	
		L	R
Player 1	U	1 , 1	2 , 0
	B	0 , 2	2 , 2

Strictly and weakly dominated strategy

- A rational player never chooses a strictly dominated strategy. Hence, any strictly dominated strategy can be eliminated.
- A rational player may choose a weakly dominated strategy.

Iterated elimination of strictly dominated strategies

- If a strategy is strictly dominated, eliminate it
- The size and complexity of the game is reduced
- Eliminate any strictly dominated strategies from the reduced game
- Continue doing so successively

Iterated elimination of strictly dominated strategies: an example

		Player 2		
		Left	Middle	Right
Player 1	Up	1 , 0	1 , 2	0 , 1
	Down	0 , 3	0 , 1	2 , 0

		Player 2	
		Left	Middle
Player 1	Up	1 , 0	1 , 2
	Down	0 , 3	0 , 1

Example: Tourists & Natives

- Only two bars (bar 1, bar 2) in a city
- Can charge price of \$2, \$4, or \$5
- 6000 tourists pick a bar randomly
- 4000 natives select the lowest price bar

- Example 1: Both charge \$2
 - each gets 5,000 customers and \$10,000
- Example 2: Bar 1 charges \$4, Bar 2 charges \$5
 - Bar 1 gets $3000+4000=7,000$ customers and \$28,000
 - Bar 2 gets 3000 customers and \$15,000

Example: Tourists & Natives

Bar 2

		\$2		\$4		\$5	
Bar 1	\$2	10	10	14	12	14	15
	\$4	12	14	20	20	28	15
	\$5	15	14	15	28	25	25

Payoffs are in thousands of dollars

Bar 2

		\$4		\$5	
Bar 1	\$4	20	20	28	15
	\$5	15	28	25	25

One More Example

- Each of n players selects a number between 0 and 100 simultaneously. Let x_i denote the number selected by player i .
- Let y denote the average of these numbers
- Player i 's payoff = $\text{abs}(x_i - 3y/5)$.

One More Example

- The normal-form representation:
 - Players: {player 1, player 2, ..., player n }
 - Strategies: $S_i = [0, 100]$, for $i = 1, 2, \dots, n$.
 - Payoff functions:

$$u_i(x_1, x_2, \dots, x_n) = \mathbf{abs}(x_i - 3y/5).$$

- Is there any dominated strategy?
- What numbers should be selected?

New solution concept: Nash equilibrium

		Player 2		
		L	C	R
Player 1	T	0 , 4	4 , 0	5 , 3
	M	4 , 0	0 , 4	5 , 3
	B	3 , 5	3 , 5	6 , 6

The combination of strategies (B, R) has the following property:

- Player 1 CANNOT do better by choosing a strategy different from B, given that player 2 chooses R.
- Player 2 CANNOT do better by choosing a strategy different from R, given that player 1 chooses B.

New solution concept: Nash equilibrium

		Player 2		
		L'	C'	R'
Player 1	T'	0 , 4	4 , 0	3 , 3
	M'	4 , 0	0 , 4	3 , 3
	B'	3 , 3	3 , 3	3.5 , 3.6

The combination of strategies (B', R') has the following property:

- Player 1 CANNOT do better by choosing a strategy different from B', given that player 2 chooses R'.
- Player 2 CANNOT do better by choosing a strategy different from R', given that player 1 chooses B'.

Nash Equilibrium: idea

- We derive a concept called a Nash Equilibrium (after the Economics Nobel Prize winning Mathematician John Nash – of “A Beautiful Mind” fame)
- **Best response:** Given a strategy for the opponent, the best response is the strategy that gives the highest payoff.
- **Nash equilibrium:** A combination of strategies such that each player plays his best response to the opponent’s best response.

Nash equilibrium (NE)

- Nash showed for the first time in his dissertation, *Non-cooperative games* (1950), that NE (in mixed strategies) **must exist** for all finite games with any number of players.
- What is NE?
 - Informally, a set of strategies is a NE if
 - No player can do better by unilaterally changing his or her strategy.
 - One can imagine that each player is told the strategies of the other players, if any player would want to do something different after being informed about the other's strategies, then that set of strategies is not NE, o.w. it is NE.

A Beautiful Mind

- IMDB Link:
 - <http://www.imdb.com/title/tt0268978/>
- Wikipedia Link:
 - http://en.wikipedia.org/wiki/A_Beautiful_Mind_%28film%29
- YouTube:
 - <http://www.youtube.com/watch?v=BrrleHl9qfs&feature=related>



Definition: Nash Equilibrium

In the normal-form game $\{S_1, S_2, \dots, S_n, u_1, u_2, \dots, u_n\}$, a combination of strategies (s_1^*, \dots, s_n^*) is a *Nash equilibrium* if, for every player i ,

$$u_i(s_1^*, \dots, s_{i-1}^*, s_i^*, s_{i+1}^*, \dots, s_n^*) \geq u_i(s_1^*, \dots, s_{i-1}^*, s_i, s_{i+1}^*, \dots, s_n^*)$$

Given others' choices, player i cannot be better-off if she deviates from s_i^*

for all $s_i \in S_i$. That is, s_i^* solves

Maximize $u_i(s_1^*, \dots, s_{i-1}^*, s_i^*, s_{i+1}^*, \dots, s_n^*)$

Subject to $s_i \in S_i$

		Prisoner 2	
		Mum	Confess
Prisoner 1	Mum	-1 , -1	-9 , 0
	Confess	0 , -9	-6 , -6

2-player game with finite strategies

- $S_1 = \{s_{11}, s_{12}, s_{13}\}$ $S_2 = \{s_{21}, s_{22}\}$
- (s_{11}, s_{21}) is a Nash equilibrium if
 - $u_1(s_{11}, s_{21}) \geq u_1(s_{12}, s_{21})$,
 - $u_1(s_{11}, s_{21}) \geq u_1(s_{13}, s_{21})$ and
 - $u_2(s_{11}, s_{21}) \geq u_2(s_{11}, s_{22})$.

Player 2

		Player 2	
		s_{21}	s_{22}
Player 1	s_{11}	$u_1(s_{11}, s_{21}), u_2(s_{11}, s_{21})$	$u_1(s_{11}, s_{22}), u_2(s_{11}, s_{22})$
	s_{12}	$u_1(s_{12}, s_{21}), u_2(s_{12}, s_{21})$	$u_1(s_{12}, s_{22}), u_2(s_{12}, s_{22})$
	s_{13}	$u_1(s_{13}, s_{21}), u_2(s_{13}, s_{21})$	$u_1(s_{13}, s_{22}), u_2(s_{13}, s_{22})$

Finding a Nash equilibrium: cell-by-cell inspection

		Player 2		
		Left	Middle	Right
Player 1	Up	1 , 0	1 , 2	0 , 1
	Down	0 , 3	0 , 1	2 , 0

		Player 2	
		Left	Middle
Player 1	Up	1 , 0	1 , 2
	Down	0 , 3	0 , 1

Ex2 : War & Peace

- Players={Country A, Country B}
- Strategies={peace, war}
- Payoffs (dollars):
 - If both Peace – reap benefits of Peace
 - If one country Peace and other War, Peace country destroyed
 - If both War, both bare costs of War

		B	
		Peace	War
A	Peace	10,10	-20,-5
	War	-5,-20	-10,-10

War and Peace Nash Equil.

- $BRA(BPeace) = APeace$
- $BRB(APeace) = BPeace$
- SO:
 1. If B plays Peace, A should play Peace
 2. If A plays Peace, B should play Peace (back to 1)
- When B is playing Peace A will **always** play Peace and when A is playing Peace B will **always** play Peace
- (Peace, Peace) is a Nash Equilibrium

		B ↓ 1	
		Peace	War
A → 2	Peace	10,10	-20,-5
	War	-5,-20	-10,-10

War and Peace Nash Equil.

- $BRA(BWar) = AWar$
- $BRB(AWar) = BWar$
- SO:
 1. If B plays War, A should play War
 2. If A plays War, B should play War (back to 1)
- When B is playing War A will **always** play War and when A is playing War B will **always** play War
- (War, War) is a Nash Equilibrium

		B	
			1 ↓
		Peace	War
A	Peace	10,10	-20,-5
	War	-5,-20	-10,-10
	2 →		

War and Peace Nash Equil.

- So, (Peace, Peace) and (War, War) are NE
- What about Pareto efficiency?
- The only outcome that doesn't allow anyone to be better off without anyone else being hurt is (Peace, Peace)
- Thus, (P,P) is a PE NE

		B	
		Peace	War
A	Peace	10,10	-20,-5
	War	-5,-20	-10,-10

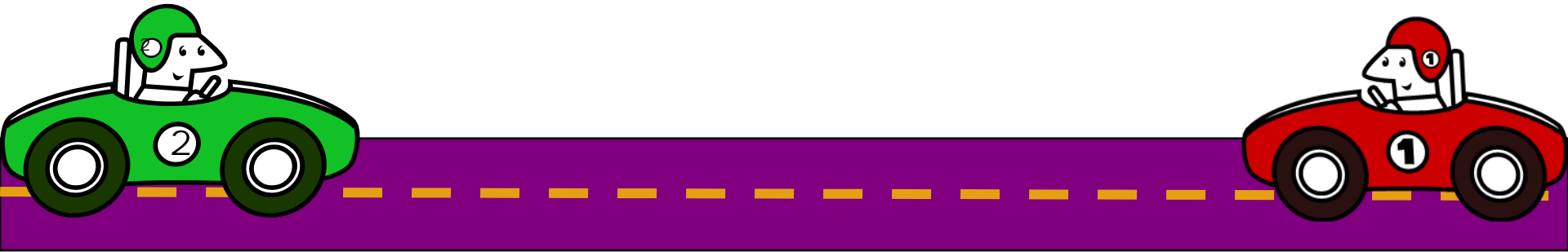
Ex3 : A game of chicken

- Players={BSB, Sky}
- Strategies={In, Out}
- Payoffs (dollars):
 - Two firms - Sky TV and British Satellite Broadcasting
 - Each consider offering satellite television services in 1990.

		BSB	
		In	Out
Sky	In	-118, -747	673, 0
	Out	0, 137	0, 0

Chicken – What it means

- Analysis of situations where conflict of interests are present



- Game of Chicken
 - driver who steers away loses
- What should drivers do?

A game of chicken

- $BRS(BIn)=SOut$
- $BRB(SOut)=BIn$
- SO:
 - If B plays In, S should play Out
 - If S plays Out, B should play In (back to 1)
- When B is playing In, S will **always** play Out and when S is playing out B will **always** play In
- (Out, In) is a Nash Equilibrium

		BSB	
		1 ↓	
		In	Out
Sky	In	-118, -747	673, 0
	2 → Out	0, 137	0, 0

A game of chicken

- $BRS(BOut)=SIn$
- $BRB(SIn)=BOut$
- SO:
 - If B plays Out, S should play In
 - If S plays In, B should play Out (back to 1)
- When B is playing Out, S will **always** play In and when S is playing In B will **always** play Out
- (In, Out) is a Nash Equilibrium

		BSB	
		In	Out
Sky	In	-118, -747	673, 0
	Out	0, 137	0, 0

An arrow labeled '1' points down from the top of the 'Out' column. An arrow labeled '2' points right from the left of the 'In' row.

A game of chicken

- So, (In, Out) and (Out, In) are NE
- What about Pareto efficiency?
- The only outcome that can be improved upon by both is (In, In) so (In, Out), (Out, In) and (Out, Out) are Pareto Optimal

		BSB	
		In	Out
Sky	In	-118,-747	673, 0
	Out	0,137	0, 0

Best response function: example

		Player 2		
		L'	C'	R'
Player 1	T'	0 , 4	4 , 0	3 , 3
	M'	4 , 0	0 , 4	3 , 3
	B'	3 , 3	3 , 3	<u>3.5</u> , <u>3.6</u>

- If Player 2 chooses L' then Player 1's best strategy is M'
- If Player 2 chooses C' then Player 1's best strategy is T'
- If Player 2 chooses R' then Player 1's best strategy is B'
- If Player 1 chooses B' then Player 2's best strategy is R'
- Best response: the best strategy one player can play, given the strategies chosen by all other players

Example: Tourists & Natives

		Bar 2		
		\$2	\$4	\$5
Bar 1	\$2	10 , 10	14 , 12	14 , 15
	\$4	12 , 14	20 , 20	28 , 15
	\$5	15 , 14	15 , 28	25 , 25

Payoffs are in thousands of dollars

- what is Bar 1's best response to Bar 2's strategy of \$2, \$4 or \$5?
- what is Bar 2's best response to Bar 1's strategy of \$2, \$4 or \$5?

2-player game with finite strategies

- $S_1 = \{s_{11}, s_{12}, s_{13}\}$ $S_2 = \{s_{21}, s_{22}\}$
- Player 1's strategy s_{11} is her best response to Player 2's strategy s_{21} if

$$u_1(s_{11}, s_{21}) \geq u_1(s_{12}, s_{21}) \text{ and}$$

$$u_1(s_{11}, s_{21}) \geq u_1(s_{13}, s_{21}).$$

		Player 2	
		s_{21}	s_{22}
Player 1	s_{11}	$u_1(s_{11}, s_{21}), u_2(s_{11}, s_{21})$	$u_1(s_{11}, s_{22}), u_2(s_{11}, s_{22})$
	s_{12}	$u_1(s_{12}, s_{21}), u_2(s_{12}, s_{21})$	$u_1(s_{12}, s_{22}), u_2(s_{12}, s_{22})$
	s_{13}	$u_1(s_{13}, s_{21}), u_2(s_{13}, s_{21})$	$u_1(s_{13}, s_{22}), u_2(s_{13}, s_{22})$

Using best response function to find Nash equilibrium

- In a 2-player game, (s_1, s_2) is a Nash equilibrium if and only if player 1's strategy s_1 is her best response to player 2's strategy s_2 , and player 2's strategy s_2 is her best response to player 1's strategy s_1 .

		Prisoner 2	
		Mum	Confess
Prisoner 1	Mum	-1 , -1	-9 , <u>0</u>
	Confess	<u>0</u> , -9	<u>-6</u> , <u>-6</u>

Using best response function to find Nash equilibrium: example

		Player 2		
		L'	C'	R'
Player 1	T'	0 , <u>4</u>	<u>4</u> , 0	3 , 3
	M'	<u>4</u> , 0	0 , <u>4</u>	3 , 3
	B'	3 , 3	3 , 3	<u>3.5</u> , <u>3.6</u>

- M' is Player 1's best response to Player 2's strategy L'
- T' is Player 1's best response to Player 2's strategy C'
- B' is Player 1's best response to Player 2's strategy R'
- L' is Player 2's best response to Player 1's strategy T'
- C' is Player 2's best response to Player 1's strategy M'
- R' is Player 2's best response to Player 1's strategy B'

Example: Tourists & Natives

		Bar 2		
		\$2	\$4	\$5
Bar 1	\$2	10 , 10	14 , 12	14 , 15
	\$4	12 , 14	20 , 20	28 , 15
	\$5	15 , 14	15 , 28	25 , 25

Payoffs are in thousands of dollars

Use best response function to find the Nash equilibrium.

Example: The battle of the sexes

		Pat	
		Opera	Prize Fight
Chris	Opera	<u>2</u> , <u>1</u>	0 , 0
	Prize Fight	0 , 0	<u>1</u> , <u>2</u>

- Opera is Player 1's best response to Player 2's strategy Opera
- Opera is Player 2's best response to Player 1's strategy Opera
 - Hence, (Opera, Opera) is a Nash equilibrium
- Fight is Player 1's best response to Player 2's strategy Fight
- Fight is Player 2's best response to Player 1's strategy Fight
 - Hence, (Fight, Fight) is a Nash equilibrium

Example: Matching pennies

		Player 2	
		Head	Tail
Player 1	Head	-1 , <u>1</u>	<u>1</u> , -1
	Tail	<u>1</u> , -1	-1 , <u>1</u>

- Head is Player 1's best response to Player 2's strategy Tail
 - Tail is Player 2's best response to Player 1's strategy Tail
 - Tail is Player 1's best response to Player 2's strategy Head
 - Head is Player 2's best response to Player 1's strategy Head
- Hence, NO Nash equilibrium

Definition: best response function

In the normal-form game

$$\{S_1, S_2, \dots, S_n, u_1, u_2, \dots, u_n\},$$

if player 1, 2, ..., $i-1$, $i+1$, ..., n choose strategies $s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n$, respectively,

Given the strategies chosen by other players

then player i 's best response function is defined by

$$B_i(s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n) =$$

$$\{s_i \in S_i : u_i(s_1, \dots, s_{i-1}, s_i, s_{i+1}, \dots, s_n)$$

$$\geq u_i(s_1, \dots, s_{i-1}, s'_i, s_{i+1}, \dots, s_n), \text{ for all } s'_i \in S_i\}$$

Player i 's best response

Definition: best response function

An alternative definition:

Player i 's strategy $s_i \in B_i(s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$ if and only if it solves (or it is an optimal solution to)

Maximize $u_i(s_1, \dots, s_{i-1}, s'_i, s_{i+1}, \dots, s_n)$

Subject to $s'_i \in S_i$

where $s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n$ are given.

Player i 's best response to other players' strategies is an optimal solution to

Using best response function to define Nash equilibrium

In the normal-form game $\{S_1, \dots, S_n, u_1, \dots, u_n\}$, a combination of strategies (s_1^*, \dots, s_n^*) is a *Nash equilibrium* if for every player i ,

$$s_i^* \in B_i(s_1^*, \dots, s_{i-1}^*, s_{i+1}^*, \dots, s_n^*)$$

- A set of strategies, one for each player, such that each player's strategy is best for her, given that all other players are playing their strategies, or
- A stable situation that no player would like to deviate if others stick to it

Nash equilibrium survive iterated elimination of strictly dominated strategies

		Player 2		
		Left	Middle	Right
Player 1	Up	<u>1</u> , 0	<u>1</u> , <u>2</u>	0 , <u>1</u>
	Down	0 , <u>3</u>	0 , 1	<u>2</u> , 0

		Player 2	
		Left	Middle
Player 1	Up	<u>1</u> , 0	<u>1</u> , <u>2</u>
	Down	0 , <u>3</u>	0 , <u>1</u>

The strategies that survive iterated elimination of strictly dominated strategies are not necessarily Nash equilibrium strategies

		Player 2		
		L'	C'	R'
Player 1	T'	0 , <u>4</u>	<u>4</u> , 0	3 , 3
	M'	<u>4</u> , 0	0 , <u>4</u>	3 , 3
	B'	3 , 3	3 , 3	<u>3.5</u> , <u>3.6</u>

Summary

- In an n -player normal-form game, if iterated elimination of strictly strategies eliminates all but the strategies $(s_1^*, s_2^*, \dots, s_n^*)$, then $(s_1^*, s_2^*, \dots, s_n^*)$ is the unique Nash equilibrium.
- In an n -player normal-form game, if the strategies $(s_1^*, s_2^*, \dots, s_n^*)$ is a Nash equilibrium then they survive iterated elimination of strictly strategies. But the strategies that survive iterated elimination of strictly dominated strategies are not necessarily are Nash equilibrium strategies.

Cournot model of duopoly

- A product is produced by only two firms: firm 1 and firm 2. The quantities are denoted by q_1 and q_2 , respectively. Each firm chooses the quantity without knowing the other firm has chosen.
- The market price is $P(Q)=a-Q$, where a is a constant number and $Q=q_1+q_2$.
- The cost to firm i of producing quantity q_i is $C_i(q_i)=cq_i$.

Cournot model of duopoly

The normal-form representation:

➤ Set of players: **{ Firm 1, Firm 2 }**

➤ Sets of strategies: $S_1=[0, +\infty), S_2=[0, +\infty)$

➤ Payoff functions:

$$u_1(q_1, q_2)=q_1(a-(q_1+q_2)-c)$$

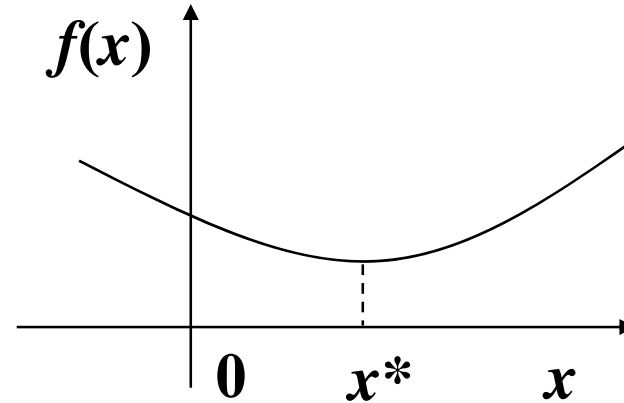
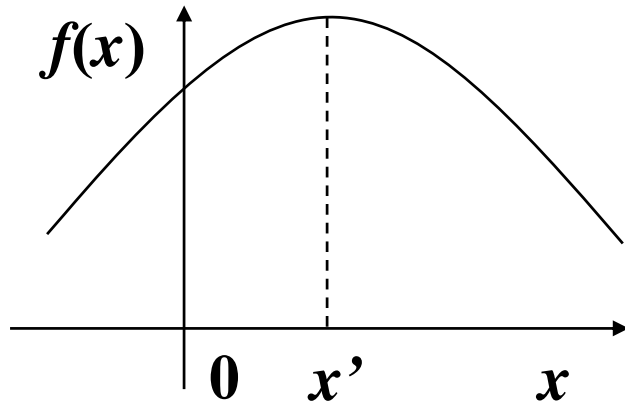
$$u_2(q_1, q_2)=q_2(a-(q_1+q_2)-c)$$

Cournot model of duopoly

- How to find a Nash equilibrium
 - Find the quantity pair (q_1^*, q_2^*) such that q_1^* is firm 1's best response to Firm 2's quantity q_2^* and q_2^* is firm 2's best response to Firm 1's quantity q_1^*
 - That is, q_1^* solves
$$\text{Max } u_1(q_1, q_2^*) = q_1(a - (q_1 + q_2^*) - c)$$
subject to $0 \leq q_1 \leq +\infty$
 - and q_2^* solves
$$\text{Max } u_2(q_1^*, q_2) = q_2(a - (q_1^* + q_2) - c)$$
subject to $0 \leq q_2 \leq +\infty$

Maximum and minimum

- **First Order Condition:** if a point x is a maximum or a minimum then x satisfies the first order condition (FOC): $f'(x) = 0$.
- If $f(x)$ is concave and x' satisfies the FOC, then x' is a maximum.
- If $f(x)$ is convex and x^* satisfies the FOC, then x^* is a minimum.



Cournot model of duopoly

- How to find a Nash equilibrium

➤ Solve

$$\text{Max } u_1(q_1, q_2^*) = q_1(a - (q_1 + q_2^*) - c)$$

$$\text{subject to } 0 \leq q_1 \leq +\infty$$

$$\text{FOC: } a - 2q_1 - q_2^* - c = 0$$

$$q_1 = (a - q_2^* - c)/2$$

Cournot model of duopoly

- How to find a Nash equilibrium

➤ Solve

$$\text{Max } u_2(q_1^*, q_2) = q_2(a - (q_1^* + q_2) - c)$$

$$\text{subject to } 0 \leq q_2 \leq +\infty$$

$$\text{FOC: } a - 2q_2 - q_1^* - c = 0$$

$$q_2 = (a - q_1^* - c)/2$$

Cournot model of duopoly

- How to find a Nash equilibrium
 - The quantity pair (q_1^*, q_2^*) is a Nash equilibrium if

$$q_1^* = (a - q_2^* - c)/2$$

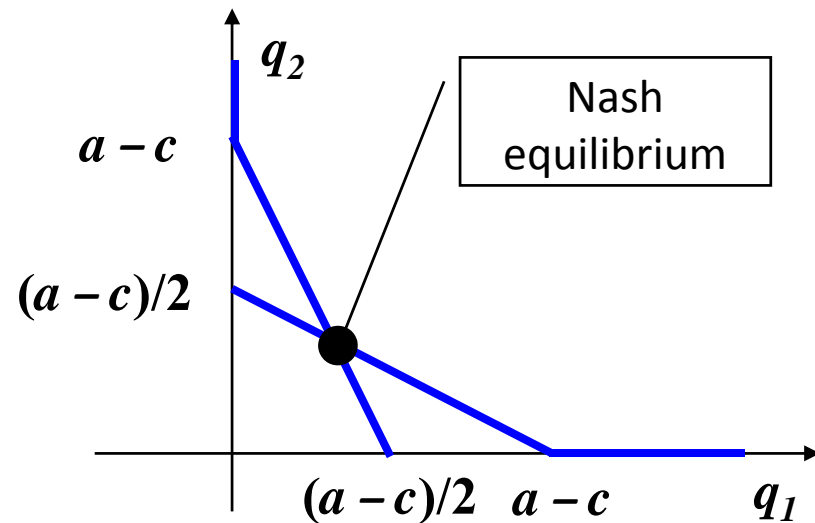
$$q_2^* = (a - q_1^* - c)/2$$

- Solving these two equations gives us

$$q_1^* = q_2^* = (a - c)/3$$

Cournot model of duopoly

- Best response function
 - Firm 1's best function to firm 2's quantity q_2 :
 $R_1(q_2) = (a - q_2 - c)/2$ if $q_2 < a - c$; 0 , otherwise
 - Firm 2's best function to firm 1's quantity q_1 :
 $R_2(q_1) = (a - q_1 - c)/2$ if $q_1 < a - c$; 0 , otherwise



Want to know more?

- Here are some resources that might be helpful:
 - <http://www.gametheory.net/>
 - M. J. Osborne, An Introduction to Game Theory
 - M. J. Osborne and A. Rubinstein, A Course in Game Theory