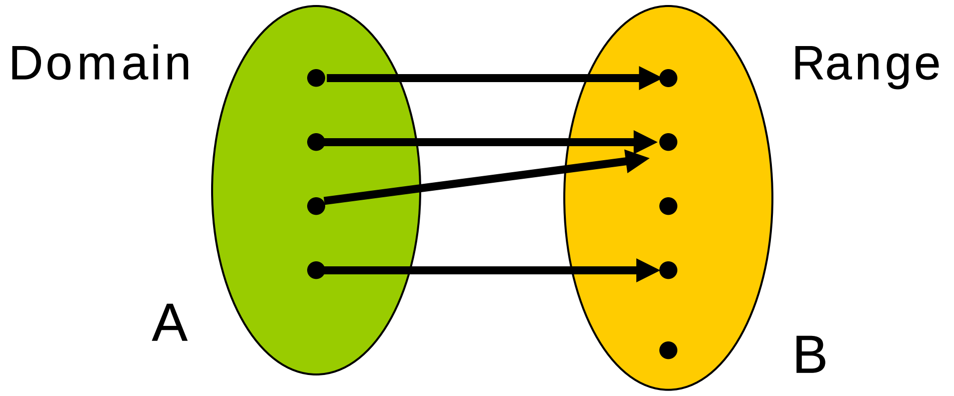
# L03: Functions and Counting

2012 Summer Supplementary Math Course,

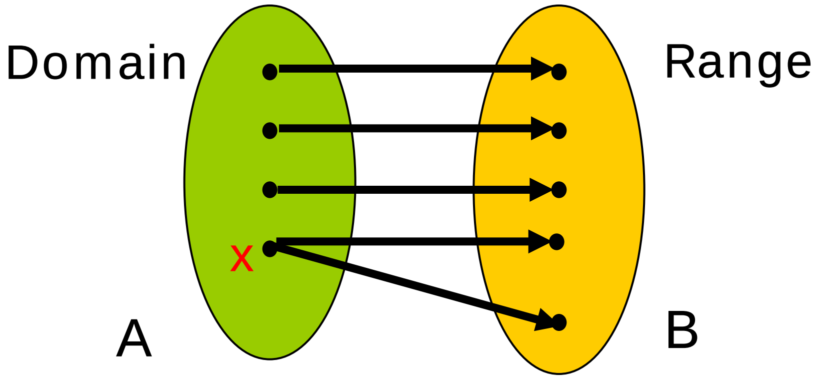
CSE Department, HKUST.

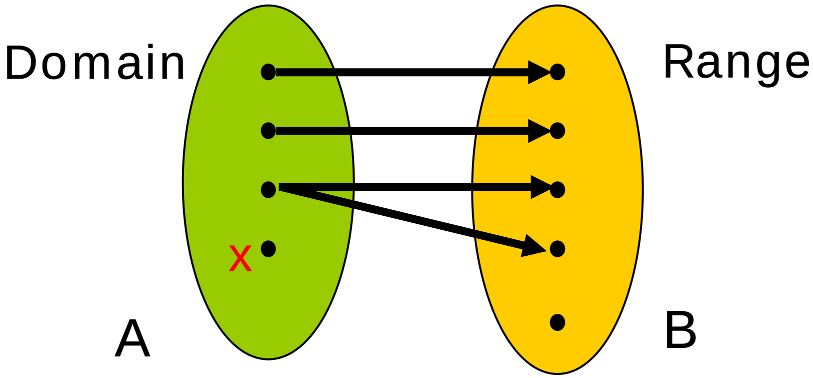
# Definition of function

- $f$ is a **function** from $A$ to $B$ ($f$: $A \rightarrow B$) if and only if, for each $x$ in $A$, there exists one and only one $y$ in $B$ such that $y=f(x)$.
- Example of a function:



# Some mappings are not functions




- *x* is mapped to more than one value in *B*.
- It is not a function!

- *x* is **not** mapped to any value in *B*.
- It is not a function!

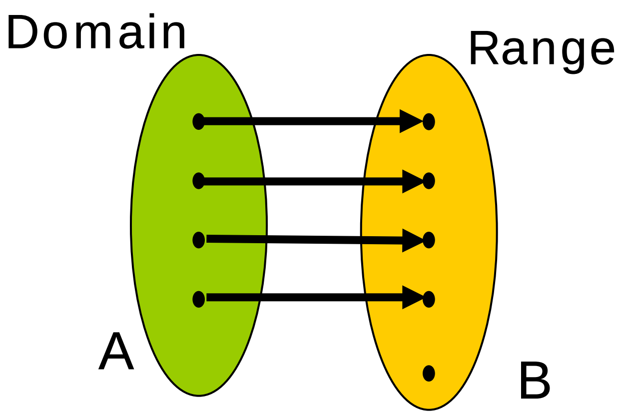
# Domain and range

- In our example:
  - *A* is called the **domain** (**pre-image**) of *f*.
  - *B* is called the **range** (**image**) of *f*.
- The definition of a function can be written as: every element in the **domain** is mapped to **exactly** one element in the **range**.
- Question: $$f:\mathbb{R}\to\mathbb{R},\quad f(x)=\frac{x^2-4}{x-2}.$$

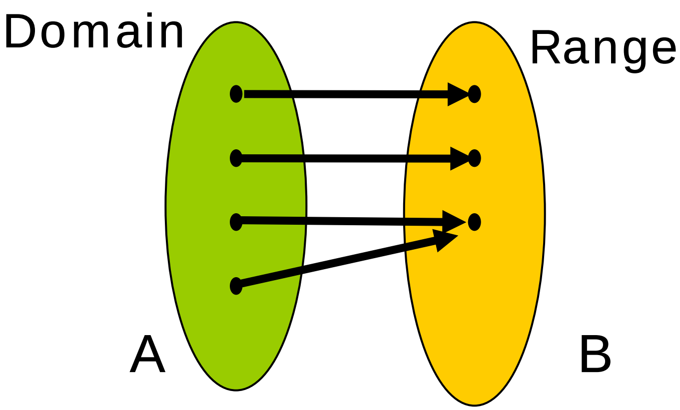
  Is *f* a function?

# Injective

- A function $f$: $A \rightarrow B$ is **injective (one to one)** if and only if, for all $a_1$, $a_2$ in $A$, $f(a_1) = f(a_2)$ implies $a_1 = a_2$.



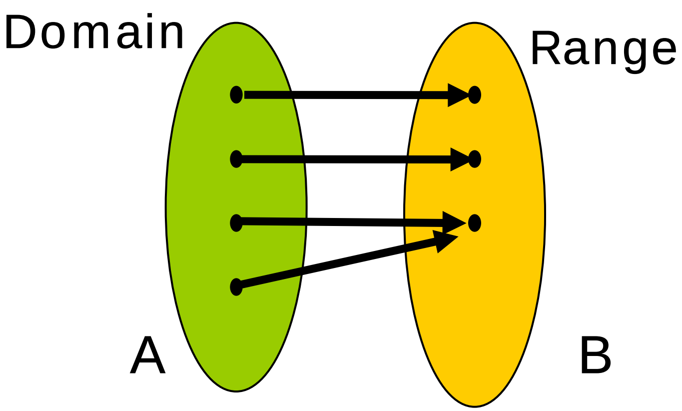

Injective



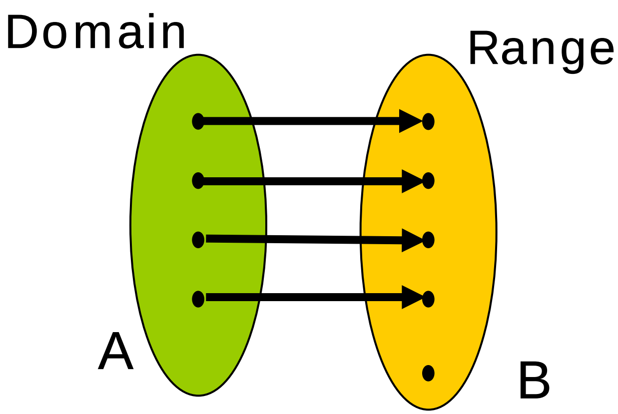

Not injective

# Surjective

- A function $f$: $A \rightarrow B$ is **surjective** (**onto**) if and only if, for all $b$ in $B$, there exists $a$ in $A$, such that $f(a)=b$.




surjective




Not surjective

# Example 1

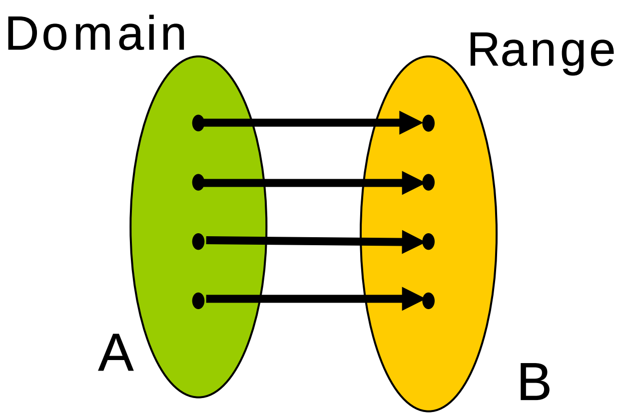
- Given $f$: $\mathbb{N} \to \mathbb{N}$ such that $f(x)=x+1$, $f$ is injective but not surjective.
  - For all $x_1$, $x_2$ in $\mathbb{N}$,
  - $f(x_1)=f(x_2) \Rightarrow x_1+1=x_2+1 \Rightarrow x_1=x_2$.
  - So $f$ is injective.
  - Assume $f$ is surjective.
  - 0 (in $\mathbb{N}$) is mapped by some $x$ (in $\mathbb{N}$).
  - There exists an $x$ in $\mathbb{N}$ such that $f(x)=0$.
  - So, $x+1=0 \Rightarrow x=-1$.
  - $x$ is **not** in $\mathbb{N}$ which leads to a contradiction.
  - Therefore, $f$ is not surjective.

# Example 2

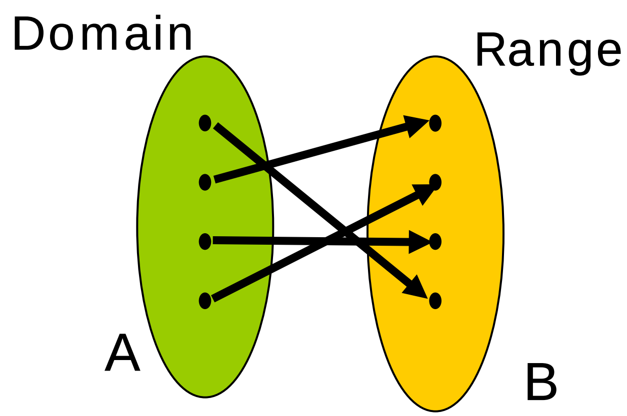
- Given $f$: $\mathbb{R} \rightarrow \mathbb{R}^+$ such that $f(x) = x^2$, $f$ is surjective but not injective.
  - For all $y$ in $\mathbb{R}^+$, there exists $x$ in $\mathbb{R}$, such that:
  - $y = x^2 = f(x)$ (we choose $x = \sqrt{y}$).
  - So $f$ is surjective.
  - Note that 1, -1 are in $\mathbb{R}$, and
  - $1^2 = (-1)^2 = 1$.
  - That is, $f(1) = f(-1)$.
  - So $f$ is not injective.

# Bijective

- A function $f$: $A \rightarrow B$ is **bijective (one-to-one correspondence)** if and only if $f$ is both injective and surjective.

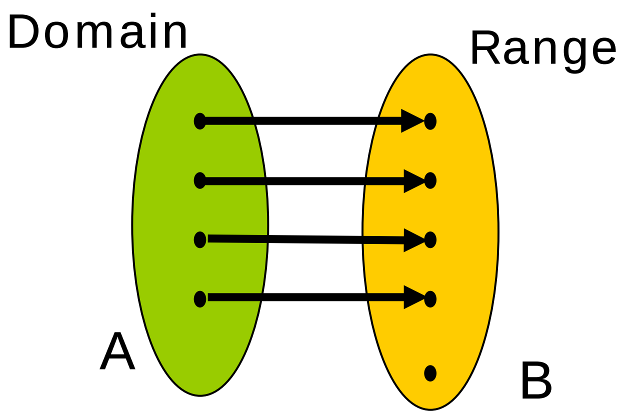



Bijective

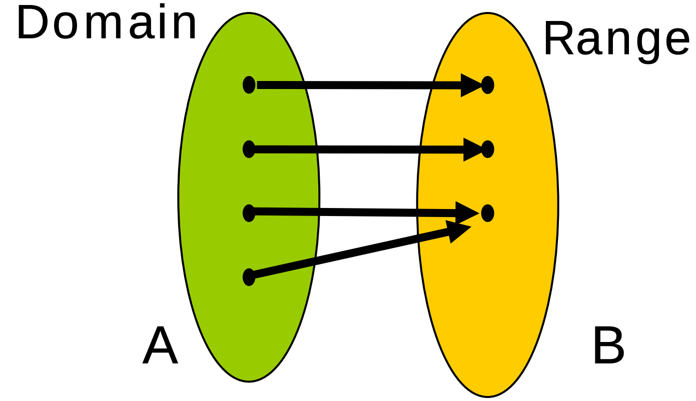



Bijective

# Bijective




Not bijective




Not bijective

- Exercise: Given $f$: $\mathbb{N} \to \mathbb{Z}^+$ such that $f(x) = x + 1$, prove that $f$ is bijective.

# Inverse function

- The inverse function of $f$ is denoted by $f^{-1}$.
- $f^{-1}$ exists if and only if $f$ is bijective.
- Given a bijective function $f: A \rightarrow B$ such that $y=f(x)$, the inverse function is $f^{-1}: B \rightarrow A$ such that $f^{-1}(y)=x$.
- Exercise: Given a function $f: A \rightarrow B$ such that $y=f(x)$, we define a mapping $m: B \rightarrow A$ such that $y$ is mapped to $x$. Explain why $m$ is not a function when $f$ is not bijective.

# Inverse function

- If $f$ is bijective, $f^{-1}$ is also bijective.
- Then the inverse of $f^{-1}$ exists.
- And $(f^{-1})^{-1} = f$.

- Exercise: Given $f$: $\mathbb{R}-\{-1\}\to\mathbb{R}-\{1\}$ such that $f(x)=(x-1)/(x+1)$. Find $f^{-1}$.

# Composite function

- Let $f$: $A \to B$ and $g$: $B \to C$ be two functions. The **composition function** $g \circ f$: $A \to C$ is a function such that $(g \circ f)(x) = g(f(x))$.
  - Let $f$: $\mathbb{R} \to \mathbb{R}^+ \cup \{0\}$ such that $f(x) = x^2$.
  - Let $g$: $\mathbb{R}^+ \cup \{0\} \to \mathbb{R}^- \cup \{0\}$ such that $g(x) = -x$.
  - Let $h = g \circ f$.
  - Then $h$: $\mathbb{R} \to \mathbb{R}^- \cup \{0\}$ is a function such that $h(x) = g(f(x)) = g(x^2) = -x^2$.

# Composite function

- Remarks:
  - $(g \circ f)(x)$ means $h(x)$ such that $h = g \circ f$.
  - $g(f(x))$ means $g(y)$ such that $y = f(x)$.

- Question: Let $f$: $\mathbb{R} \rightarrow \mathbb{R}$ such that $f(x) = x^2$, what is $f \circ f \circ f$?

# Practical use of Functions

- The “procedure” or “function” in programming use the concept of functions.
  - Return type is similar to Range
  - Parameters are similar to Domain

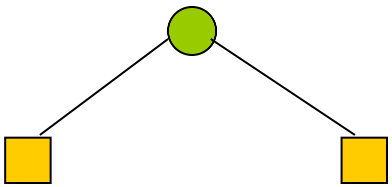
# Counting

# Multiplication principle of counting

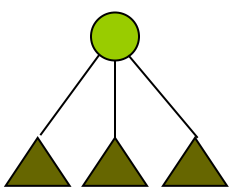
- **Multiplication principle of counting**:
  - Suppose that two tasks $T_1$ and $T_2$ are to be performed in sequence.
  - $T_1$ can be performed in $n_1$ ways.
  - *For each* of these ways, $T_2$ can be performed in $n_2$ ways.
  - Then the **sequence $T_1T_2$** can be performed in **$n_1n_2$ ways**.

# Multiplication principle of counting

- Here is an example:

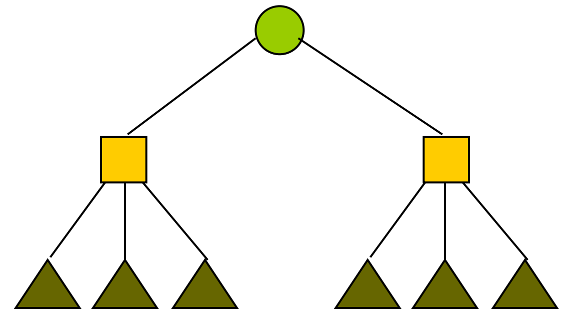


Possible ways of
performing task $T_1 = 2$



Possible ways of
performing task $T_2 = 3$

$\Rightarrow$



Possible ways of performing
task $T_1$ and then task $T_2$
$= 2 \times 3 = 6$.

# Multiplication principle of counting

- What is the total number of possible different outcomes for tossing a 6-faced dice once, and then drawing one card from a pack of 52 cards?
  - Answer: $6 \times 52 = 312$.
- What is the total number of possible combinations of an I.D. card number, with one capital letter (A-Z) at the beginning, followed by 6 digits (0-9)?
  - Answer: $26 \times 10^6 = 26000000$.

# Factorial

- The **factorial** of $n$ is denoted by $n!$, where $n! = n(n-1)(n-2)\ldots1$.
  - $1! = 1$.
  - $2! = 2(1) = 2$.
  - $3! = 3(2)(1) = 6$.
- For $n = 0$, we define $0! = 1$.
- Note that for $n \geq 1$, $n! = n(n-1)!$
- Exercise: Without using a calculator, compute the value of $5!$.

# Linear arrangements

- Given three objects: $a$, $b$ & $c$.
- The set of possible linear arrangements (permutations) of the three objects = $\{abc, acb, bac, bca, cab, cba\}$.
- The number of linear arrangements = 6.
- **Fact**: The number of linear arrangements of $n$ **distinct** objects = $n!$.
- Let's try to prove this fact by induction.

# Linear arrangements

- Let $A(n)$ = the number of linear arrangements of $n$ distinct objects.
- Note that:
  - The number of linear arrangements of $n$ distinct objects = the number of possibilities for the first position $\times$ the number of arrangements of the remaining $n$-1 objects.
  - So, $A(n)=n \times A(n-1)$.
- Let $P(n)$ be the statement "$A(n) = n!$".
- Since $A(1)=1$ and $1!=1$, $P(1)$ is true.

# Linear arrangements

- Assume $P(k)$ is true, that is: $A(\mathrm{k}) = k!$.
- When $n=k+1$,
  - $A(k+1)$

    $=(k+1)A(k)$

    $=(k+1)k!$

    $=(k+1)!$.
  - So $P(k+1)$ is true.
- By induction, $P(n)$ is true $\forall n \in \mathbb{Z}^+$.

# Question

- Question: There are 20 students in a classroom with 20 chairs. What is the total number of different sitting plans?
- Question: There are 5 distinct balls in a bag. 5 balls are drawn one-by-one in order from the bag. What is the total number of possible outcomes when
  - drawing each ball without replacement?
  - drawing each ball with replacement?

# Permutations

- The number of **permutations** of $n$ distinct objects taken $r$ at a time is denoted by ${}_nP_r$.
- ${}_nP_r = n(n-1)(n-2)\ldots(n-r+1)$.
- Idea:
  - $n$ ways for taking the 1st objects.
  - $n$-1 ways for taking the 2nd objects.
  - ...
  - $n$-($r$-1) ways for taking the rth objects.
- Using another notation: $${}_nP_r = \frac{n!}{(n-r)!}.$$

# Question

- A string of 3 **distinct** capital letters (A-Z) is generated. What is the total number of possible string patterns that can be generated?
- 7 **identical** 6-faced dices are tossed. Then, 3 dice are chosen one-by-one in order. What is the number of different outcomes?

# Distinguishable permutations

- How many permutations of letters can be formed from the word “MAST”?
  - Answer: $4!=24$.
- How many distinguishable permutations of letters can be formed from the word “MASS”?
  - Answer: $4!/2!=24/2=12$.
- Reason:
  - When counting the number as $4!$, the two “S” are treated as different letters. For each outcome in those $4!$ outcomes, it appears $2!$ times.

# Distinguishable permutations

- **Theorem**:
  - Given a collection of $n$ objects, where the objects belong to $r$ distinct classes, such that
  - there are $k_1$ objects of the 1st class,
  - there are $k_2$ objects of the 2nd class,
  - ......
  - there are $k_r$ objects of the $r$th class.
  - Note that $k_1 + k_2 + \ldots + k_r = n$.
  - Then, the total number of **distinguishable permutations** of the $n$ objects $= \dfrac{n!}{k_1!k_2!\cdots k_r!}$.

# Distinguishable permutations

- How many distinguishable permutations of letters can be formed form the word "BANANA"?
  - Answer: 6!/(1!3!2!)=60.
- There are 3 red balls, 2 yellow balls, 4 blue balls, 2 white balls. What is the total number of possible distinguishable linear arrangements of the 11 balls?
  - Answer: 11!/(3!2!4!2!)=69300.

# Combinations

- The number of combinations of *n* distinct objects taken *r* at a time is denoted by ${}_nC_r$.
- We have:

$${}_nC_r = \frac{n!}{r!(n-r)!}.$$

- Combinations: the order of the *r* objects does **NOT** matter.
- Permutations: each **different order** of the *r* objects is treated as a **different outcome**.

# Combinations

- ${}_nC_r$ is said to be the number of $r$-element subsets of an $n$-element set.
- Proof of the formula for ${}_nC_r$:
  - Note: number of $r$-element subsets from an $n$-element set $\times$ number of permutations of the selected $r$-element subset = number of permutations of $n$ objects taken $r$ at a time.
  - So, ${}_nC_r \times r! = {}_nP_r$

$$\Rightarrow {}_nC_r = \frac{n!}{r!(n-r)!}.$$

# Combinations

- What is the total number of possible combinations of the six numbers (excluding the special number) in a Mark Six draw?
  - Answer: ${}_{47}C_6 = 10737573$.
- In how many ways can a committee of three teachers and two students be selected from seven teachers and eight students respectively?
  - Answer: ${}_7C_3 \times {}_8C_2 = 980$.

# Practical use of counting

- Allows you to roughly estimate size and possible combination of input to a program or a function.
  - E.g. there are 2x2=4 possible combinations for a condition with two boolean variables
- Important for calculating probabilities.