Improving Dynamic Programming

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Last Updated 25/08/2010
Dynamic Programming

DP creates a search space and calculates optimal cost for every item in the search space.

Optimal cost of larger items is based on optimal cost of smaller items.

Final Result: usually cost of largest item in search space.

Running time of DP algorithm, is time required to calculate all costs.
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Chain Matrix Multiplication: Finding “cheapest” way to multiply matrices $A_1, \ldots, A_n$ where $A_i$ is a $p_{i-1} \times p_i$ matrix.

$$m[i, j] = \begin{cases} 
0 & \text{if } i = j \\
\min_{i \leq k < j} m[i, k] + m[k + 1, j] + p_{i-1} p_k p_j & \text{if } i > j
\end{cases}$$

$m[i, j]$ is “best” way of multiplying $A_i, \ldots, A_j$
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Want $m[1, n]$ and corresponding set of multiplications
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Longest Common Subsequence: Find LCS of strings

\[ X = \langle x_1, \ldots, x_m \rangle, \ Y = \langle y_1, \ldots, y_n \rangle. \]

\[
c[i, j] = \begin{cases} 
0 & \text{if } i = 0 \text{ or } j = 0 \\
\text{c}[i - 1, j - 1] + 1 & \text{if } i, j > 0 \text{ and } x_i = x_j \\
\text{max}(\text{c}[i - 1, j], \text{c}[i, j - 1]) & \text{if } i, j > 0 \text{ and } x_i \neq x_j
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\( c[i, j] \) is length of LCS of \( \langle x_1, \ldots, x_i \rangle, \langle y_1, \ldots, y_j \rangle \).
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Want \(c[m, n]\) and corresponding LCS.
DP is taught to all CS undergrads.

Isn’t it well understood? What’s left to do?
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A LOT!:
There are techniques for *speeding up* DP computations by an order of magnitude.
Also techniques for *reducing space requirements* by an order of magnitude
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New speedups are still being found, still on ad-hoc basis. Crying need for a general theory of speedups, that can be referenced by application users.
In this talk, will combine

- one well-known time speedup: Monge Property + SMAWK algorithm and

- one basic $\Theta(n)$ space improvement (Hirschberg 1975)
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More details in:

A Dynamic Programming Approach to Length-Limited Huffman Coding: Space Reduction With the Monge Property
M. Golin & Y. Zhang

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\[(a) \quad H(i) = \min_{0 \leq j < i} \left( H(j) + w(j, i) \right) \]

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\[ 0 \leq i \leq n \quad \Theta(n^2) \text{ time} \]

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\[ 0 \leq i \leq n \quad n^2 \rightarrow n \]
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Calculating \( H(n, D) \) requires only \( O(n) \) space.

Note that storing the table uses \( \Theta(Dn) \) space, where \( D \) could be quite large.

Naive method of constructing solution from DP table, requires backtracking through table requires storing entire DP table \( \Rightarrow \Theta(Dn) \) space.
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Will see how to reduce this to \( O(n) \) space.
Outline

• The Monge Speedup

• Saving Space While Saving Time
The Monge Speedup

- $M$ is an $m \times n$ matrix
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- $\text{RM}_M(i)$ is column index of (rightmost) min item on row $i$ of $M$.
- $M$ is Monotone if $\forall i \leq i', \text{RM}_M(i) \leq \text{RM}_M(i')$. 
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\[
RM_M(1) = 2 \\
RM_M(2) = 4 \\
RM_M(3) = 4 \\
RM_M(4) = 4 \\
RM_M(5) = 6 \\
RM_M(6) = 6
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- $M$ is Monotone if $\forall i \leq i', \ RM_M(i) \leq RM_M(i')$.
- $2 \times 2$ monotone matrices have form
  
  $\begin{bmatrix} 2 & 4 \\ 4 & 5 \end{bmatrix}$  
  $\begin{bmatrix} 2 & 3 \\ 5 & 3 \end{bmatrix}$  
  $\begin{bmatrix} 7 & 1 \\ 2 & 2 \end{bmatrix}$

- An $m \times n$ matrix $M$ is Totally Monotone (TM) if every $2 \times 2$ submatrix is Monotone.

  (submatrix: not necessarily contiguous in the original matrix)
The SMAWK Algorithm

- Motivation:
  Find all $m$ row minima of an implicitly given $m \times n$ matrix $M$
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- SMAWK Algorithm
  [Aggarwal, Klawe, Moran, Shor, Wilber (1986)]
  - If $M$ is Totally Monotone,
    all $m$ row minima can be found in $O(m + n)$ time.
  - Usually $m = \Theta(n)$
    $\Theta(n)$ speedup: $O(n^2)$ down to $O(n)$.

- See http://www.cs.ust.hk/mjg_lib/Classes/COMP572_Fall07/Notes/SMAWK.pdf for proof
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• SMAWK was culmination of decade(s) of work on similar problems;
  speedups using convexity and concavity.
  Has been used to speed up many DP problems, e.g., computational geometry, bioinformatics, $k$-center on a line, etc.
The Monge Property

- **Motivation:** TM is often established via Monge property
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• $m \times n$ matrix $M$ is Monge if $\forall i \leq i'$ and $\forall j \leq j'$

\[ M_{i,j} + M_{i',j'} \leq M_{i',j} + M_{i,j'} \]
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• $M$ is Monge $\Rightarrow$ $M$ is Totally Monotone

• Also, if $\forall i, j$, $M_{i,j} + M_{i+1,j+1} \leq M_{i+1,j} + M_{i,j+1}$, $\Rightarrow M$ is Monge.
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• Also, if $\forall i, j$, $M_{i,j} + M_{i+1,j+1} \leq M_{i+1,j} + M_{i,j+1}$, $\Rightarrow M$ is Monge.

• $\Rightarrow$ Only need to prove Monge property for adjacent rows and columns.
An Example of a Monge Matrix
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From http://en.wikipedia.org/wiki/Monge_array

\[
\begin{bmatrix}
10 & 17 & 13 & 28 & 23 \\
17 & 22 & 16 & 29 & 23 \\
24 & 28 & 22 & 34 & 24 \\
11 & 13 & 6 & 17 & 7 \\
45 & 44 & 32 & 37 & 23 \\
36 & 33 & 19 & 21 & 6 \\
75 & 66 & 51 & 53 & 34
\end{bmatrix}
\]

To see that it’s Monge, only need to check the 24 instances of

\[M_{i,j} + M_{i+1,j+1} \leq M_{i+1,j} + M_{i,j+1}\]
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Monge (or Total Monotonicity) seems an esoteric condition. In reality, it occurs very often.

Finding row minima can be used as a DP primitive.
\( \Rightarrow \) the SMAWK algorithm can be used to speed up many DPs.
Using The Monge Property

Suppose we are given DP \((H(i, 0) \text{ known, } i \leq n, \ d \leq D)\):

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H(i, d) = \min_{0 \leq j < i} \left( H(j, d - 1) + w(j, i) \right)
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To calculate \(H(\ast, d)\), simply find row-minima in \(M\)
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Given \(H(\ast, d - 1)\), SMAWK finds all \(H(\ast, d)\) in \(O(n)\) time;
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H(*, 0) \xRightarrow{O(n)} H(*, 1) \xRightarrow{O(n)} H(*, 2) \xRightarrow{O(n)} \ldots \xRightarrow{O(n)} H(*, d)
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H(\ast, 0) \xrightarrow{O(n)} H(\ast, 1) \xrightarrow{O(n)} H(\ast, 2) \xrightarrow{O(n)} \cdots \xrightarrow{O(n)} H(\ast, d)
\]

So, \(O(Dn)\) time to calculate \(H(n, d)\) and we are done!
Examples of $i \leq n$, $d \leq D$

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- **Length Limited Huffman Codes** $0 \leq p_1 \leq p_2 \leq \cdots \leq p_n$

  \[ w(j, i) = S_{2j-i} \text{ where } S_k = \sum_{i=1}^{k} p_i. \]

  $H(n-1, D)$ is cost of min-cost $D$-limited code
Examples of $i \leq n$, $d \leq D$

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- Wireless mobile paging  
  $p_1 \geq p_2 \geq \cdots \geq p_n \geq 0$

  $w(j, i) = i \left( \sum_{\ell=j+1}^{i} p_\ell \right)$

  $H(n, D)$ is min expected bandwidth required to page all items using $\leq D$ paging rounds
• $D$-Medians on a Directed Line

\[ d_1 \quad d_2 \quad d_3 \quad \ldots \quad d_{n-1} \]

\[ w_1 \quad w_2 \quad w_3 \quad w_4 \quad \ldots \quad w_{n-1} \quad w_n \]
Identify $D$ nodes as service centers.

Nodes can only be serviced by node to their left (or themselves) so node 1 must be a service center.

*Cost* of servicing request $w_i$, is $w_i$ times distance from node $i$ to nearest service center.

**Problem is to find location of $D$ service centers that minimize total service cost.**
Let $H(i, d)$ be cost of servicing nodes $[1, i]$ using exactly $d$ servers.

$$H(i, d) = \begin{cases} 
0 & n = d \\
 w(0, i) & d = 0, \ i \geq 1 \\
\min_{d-1 \leq j < i} (H(j, d - 1) + w(j, i)) & 1 \leq d < n 
\end{cases}$$

$$w(j, i) = \sum_{l=j+1}^{i} w_l(v_l - v_{j+1}), \quad v_k = \sum_{j=1}^{k-1} d_j$$
Examples of $i \leq n$, $d \leq D$

\[ H(i, d) = \min_{0 \leq j < i} \left( H(j, d - 1) + w(j, i) \right) \]

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  \[ w(j, i) = i \left( \sum_{\ell=j+1}^{i} p_{\ell} \right) \]

- **$D$-Medians on a Directed Line**
  \[ w(j, i) = \sum_{\ell=j+1}^{i} w_{\ell} (v_{\ell} - v_{j+1}) \]
Examples of \( i \leq n, \; d \leq D \)

\[
H(i, d) = \min_{0 \leq j < i} \left( H(j, d - 1) + w(j, i) \right)
\]

• Length Limited Huffman Codes

\[
w(j, i) = S_{2j - i} \quad \text{where} \quad S_k = \sum_{i=1}^{k} p_i.
\]

• Wireless mobile paging

\[
w(j, i) = i \left( \sum_{\ell=j+1}^{i} p_{\ell} \right)
\]

• \( D \)-Medians on a Directed Line

\[
w(j, i) = \sum_{l=j+1}^{i} w_l (v_l - v_{j+1})
\]

All these \( w(j, i) = w_{j,i} \) satisfy Monge property

\[
w_{j,i} + w_{j+1,i+1} \leq w_{j,i+1} + w_{j+1,i}
\]

\( \Rightarrow \) \( H(n, D) \) can be calculated in \( O(nD) \) time
Outline

• Review of the Monge Speedup

• Saving Space While Saving Time
Given a DP in the form

$$H(i, d) = \min_{0 \leq j < i} \left( H(j, d - 1) + w(j, i) \right)$$

where $w(j, i)$ are Monge, e.g., $D$-limited Huffman Encoding, $D$-Median on a line or Wireless Paging, the $H(\cdot, \cdot)$ table can be filled in using only $O(nD)$ time.
Given a DP in the form

\[ H(i, d) = \min_{0 \leq j < i} \left( H(j, d - 1) + w(j, i) \right) \quad 0 \leq i \leq n \]
\[ 0 \leq d \leq D \]

in which, the \( w(j, i) \) are Monge, e.g., \( D \)-limited Huffman Encoding, \( D \)-Median on a line or Wireless Paging, the \( H(\cdot, \cdot) \) table can be filled in using only \( O(nD) \) time.

Furthermore, calculation of \( H(\cdot, d) \) only requires knowledge of \( H(\cdot, d - 1) \). So, if \( H(n, D) \) is final goal, we can fill in table iteratively, for \( d = 1, 2, \ldots, D \), using only \( O(n) \) space.

On the other hand, finding actual “solution path” of DP, corresponding to min-cost tree, median locations or paging schedule, requires backtracking through DP table. This implies storing entire table, using \( \Theta(nD) \) space.
$H(i, d) = \min_{0 \leq j < i} \left( H(j, d - 1) + w(j, i) \right)$

$0 \leq i \leq n$

$0 \leq d \leq D$

* $w(j, i) = S_{2j-i}$ where $S_k = \sum_{i=1}^{k} p_i$. (*)
**Context:**

\[ H(i, d) = \min_{0 \leq j < i} \left( H(j, d - 1) + w(j, i) \right) \quad 0 \leq i \leq n \]

\[ 0 \leq d \leq D \]

\[ D \]-Length-Limited Huffman Coding

\[ (*) \quad w(j, i) = S_{2j-i} \quad \text{where} \quad S_k = \sum_{i=1}^{k} p_i. \]

Larmore & Hirschberg ('90) \( O(nD) \) time, \( O(n) \) space.

Very clever special-purpose algorithm; culmination of a long series of papers by various authors on this problem.
Context:

\[
H(i, d) = \min_{0 \leq j < i} \left( H(j, d - 1) + w(j, i) \right)
\]

\(0 \leq i \leq n\)

\(0 \leq d \leq D\)

**D-Length-Limited Huffman Coding**

\((*)\) \(w(j, i) = S_{2j-i}\) where \(S_k = \sum_{i=1}^{k} p_i\).

Larmore & Hirschberg ('90) \(O(nD)\) time, \(O(n)\) space.

Very clever special-purpose algorithm; culmination of a long series of papers by various authors on this problem.

Larmore & Przytycka ('91) Derived (*) DP formulation

Easy \(O(nD)\) time (Monge) algorithm but not interesting since it requires \(\Theta(nD)\) space as well.
**Context:**

$$H(i, d) = \min_{0 \leq j < i} \left( H(j, d - 1) + w(j, i) \right)$$

$$0 \leq i \leq n, 0 \leq d \leq D$$

**D-Length-Limited Huffman Coding**

\((*)\) \(w(j, i) = S_{2j-i}\) where \(S_k = \sum_{i=1}^{k} p_i\).

Larmore & Hirschberg ('90) \(O(nD)\) time, \(O(n)\) space.

Very clever **special-purpose** algorithm; culmination of a long series of papers by various authors on this problem.

Larmore & Przytycka ('91) Derived (\(\ast\)) DP formulation

Easy \(O(nD)\) time (Monge) algorithm but not interesting since it requires \(\Theta(nD)\) space as well.

Would like to reduce space for (\(\ast\)) down to \(\Theta(n)\)
\[ H(i, d) = \min_{0 \leq j < i} \left( H(j, d - 1) + w(j, i) \right) \quad \text{for} \quad 0 \leq i \leq n, \quad 0 \leq d \leq D \]
\[ H(i, d) = \min_{0 \leq j < i} \left( H(j, d - 1) + w(j, i) \right) \quad \begin{array}{l} 0 \leq i \leq n \\ 0 \leq d \leq D \end{array} \]

**Alternative Interpretation:**

Consider a layered graph in which edges only go down one level and to the right.
\[ H(i, d) = \min_{0 \leq j < i} \left( H(j, d - 1) + w(j, i) \right) \quad 0 \leq i \leq n, \quad 0 \leq d \leq D \]

**Alternative Interpretation:**

Consider a layered graph in which edges only go down one level and to the right.

\[ w\left( (d - 1, j) \rightarrow (d, i) \right) = w(j, i) \]
$H(i, d) = \min_{0 \leq j < i} \left( H(j, d - 1) + w(j, i) \right)$

$0 \leq i \leq n$

$0 \leq d \leq D$

Alternative Interpretation:

Consider a layered graph in which edges only go down one level and to the right.

$\ w((d - 1, j) \rightarrow (d, i)) = w(j, i)$

$H(i, d) =$ cost of min-cost path from $(0, 0)$ to $(d, i)$.

Given row $H(\cdot, d - 1)$, SMAWK calculates row $H(\cdot, d)$ in $O(n)$ time. By throwing away unneeded rows, can calculate $H(\cdot, D)$ in $O(nD)$ time and $O(D)$ space.
\[ H(i, d) = \min_{0 \leq j < i} \left( H(j, d - 1) + w(j, i) \right) \quad 0 \leq i \leq n \quad 0 \leq d \leq D \]

Alternative Interpretation:

Consider a layered graph in which edges only go down one level and to the right.

\[ w( (d - 1, j) \rightarrow (d, i) ) = w(j, i) \]

\[ H(i, d) = \text{cost of min-cost path from } (0, 0) \text{ to } (d, i). \]

Given row \( H(\cdot, d - 1) \), SMAWK calculates row \( H(\cdot, d) \) in \( O(n) \) time. By throwing away unneeded rows, can calculate \( H(\cdot, D) \) in \( O(nD) \) time and \( O(D) \) space.
\[ H(i, d) = \min_{0 \leq j < i} \left( H(j, d - 1) + w(j, i) \right) \quad 0 \leq i \leq n \]

\[ 0 \leq d \leq D \]

Alternative Interpretation:

Consider a layered graph in which edges only go down one level and to the right.

\[ w((d - 1, j) \rightarrow (d, i)) = w(j, i) \]

\[ H(i, d) = \text{cost of min-cost path from } (0, 0) \text{ to } (d, i). \]

Given row \( H(\cdot, d - 1) \), SMAWK calculates row \( H(\cdot, d) \) in \( O(n) \) time. By throwing away unneeded rows, can calculate \( H(\cdot, D) \) in \( O(nD) \) time and \( O(D) \) space.
\[ H(i, d) = \min_{0 \leq j < i} \left( H(j, d - 1) + w(j, i) \right) \quad \text{for} \quad 0 \leq i \leq n \quad 0 \leq d \leq D \]

**Alternative Interpretation:**

Consider a layered graph in which edges only go down one level and to the right.

\[ w((d - 1, j) \to (d, i)) = w(j, i) \]

\( H(i, d) \) = cost of min-cost path from \((0, 0)\) to \((d, i)\).

Given row \( H(\cdot, d - 1) \), SMAWK calculates row \( H(\cdot, d) \) in \( O(n) \) time. By throwing away unneeded rows, can calculate \( H(\cdot, D) \) in \( O(nD) \) time and \( O(D) \) space.
\[ H(i, d) = \min_{0 \leq j < i} \left( H(j, d - 1) + w(j, i) \right) \quad \text{for} \quad 0 \leq i \leq n, 0 \leq d \leq D \]

**Alternative Interpretation:**

Consider a layered graph in which edges only go down one level and to the right.

\[ w\left( (d - 1, j) \rightarrow (d, i) \right) = w(j, i) \]

\[ H(i, d) = \text{cost of min-cost path from (0, 0) to (d, i)}. \]

Given row \( H(\cdot, d - 1) \), SMAWK calculates row \( H(\cdot, d) \) in \( O(n) \) time. By throwing away unneeded rows, can calculate \( H(\cdot, D) \) in \( O(nD) \) time and \( O(D) \) space.
\[
H(i, d) = \min_{0 \leq j < i} \left( H(j, d - 1) + w(j, i) \right) \quad \begin{cases} 
0 \leq i \leq n \\
0 \leq d \leq D
\end{cases}
\]

Alternative Interpretation:

Consider a layered graph in which edges only go down one level and to the right.

\[
w((d - 1, j) \rightarrow (d, i)) = w(j, i)
\]

\[
H(i, d) = \text{cost of min-cost path from } (0, 0) \text{ to } (d, i).
\]

Given row \( H(\cdot, d - 1) \), SMAWK calculates row \( H(\cdot, d) \) in \( O(n) \) time. By throwing away unneeded rows, can calculate \( H(\cdot, D) \) in \( O(nD) \) time and \( O(D) \) space.
\[ H(i, d) = \min_{0 \leq j < i} \left( H(j, d - 1) + w(j, i) \right) \quad 0 \leq i \leq n \quad 0 \leq d \leq D \]

Alternative Interpretation:

Consider a layered graph in which edges only go down one level and to the right.

\[ w((d - 1, j) \rightarrow (d, i)) = w(j, i) \]

\[ H(i, d) = \text{cost of min-cost path from } (0, 0) \text{ to } (d, i). \]

Given row \( H(\cdot, d - 1) \), SMAWK calculates row \( H(\cdot, d) \) in \( O(n) \) time. By throwing away unneeded rows, can calculate \( H(\cdot, D) \) in \( O(nD) \) time and \( O(D) \) space.

On the other hand, finding optimal path to \( H(D, n) \) requires keeping entire \( \Theta(nD) \) space table to backtrack through
\[ H(i, d) = \min_{0 \leq j < i} \left( H(j, d - 1) + w(j, i) \right) \quad 0 \leq i \leq n \quad 0 \leq d \leq D \]

We will now see how to find path using \( O(D + n) \) space.

Modification of idea due to

Hirschberg ('75)

Munro & Ramirez ('82)
\[ H(i, d) = \min_{0 \leq j < i} \left( H(j, d - 1) + w(j, i) \right) \quad 0 \leq i \leq n \]
\[ 0 \leq d \leq D \]

We will now see how to find path using \( O(D + n) \) space.

Modification of idea due to

Hirschberg ('75)
Munro & Ramirez ('82)

Let \( y \) be below and to the right of \( x \).
Assume existence of an oracle \( \text{Mid}(x, y) \) that returns a midpoint (hop distance) on some min-cost \( x-y \) path.
$\text{Mid}(x, y)$ returns a midpoint (hop distance) on some min-cost $x$-$y$ path.
Mid\((x, y)\) returns a midpoint (hop distance) on some min-cost \(x-y\) path.

We now have a simple recursive procedure for building min-cost path

\textbf{Buildpath}(x,y)

\begin{itemize}
\item If \(y_d = x_{d+1}\)
  \begin{itemize}
  \item return \((x \rightarrow y)\)
  \end{itemize}
\item else
  \begin{itemize}
  \item \(z = \text{Mid}(x, y)\)
  \item Buildpath(x,z)
  \item Buildpath(z,y)
  \end{itemize}
\end{itemize}
Mid\( (x, y) \) returns a midpoint (hop distance) on some min-cost \( x-y \) path.

We now have a simple recursive procedure for building min-cost path

**Buildpath**\((x, y)\)

- If \( y_d = x_{d+1} \)
  - return \((x \rightarrow y)\)
- else
  - \( z = \text{Mid}(x, y) \)
  - Buildpath\((x, z)\)
  - Buildpath\((z, y)\)
Mid\((x, y)\) returns a midpoint (hop distance) on some min-cost \(x-y\) path.

We now have a simple recursive procedure for building min-cost path

\[\textbf{Buildpath}(x,y)\]

If \(y_d = x_{d+1}\)

return \((x \rightarrow y)\)

else

\(z = Mid(x, y)\)

Buildpath\((x,z)\)

Buildpath\((z,y)\)
Mid(x, y) returns a midpoint (hop distance) on some min-cost x-y path.

We now have a simple recursive procedure for building min-cost path

**Buildpath(x,y)**

1. If \( y_d = x_{d+1} \)
   
   return \((x \rightarrow y)\)

2. else

   \( z = \text{Mid}(x, y) \)

   Buildpath(x,z)

   Buildpath(z,y)
Mid(x, y) returns a midpoint (hop distance) on some min-cost x-y path.

We now have a simple recursive procedure for building min-cost path

\[
\text{Buildpath}(x,y) \\
\text{If } y_{d} = x_{d+1} \\
\quad \text{return } (x \rightarrow y) \\
\text{else} \\
\quad z = \text{Mid}(x, y) \\
\quad \text{Buildpath}(x,z) \\
\quad \text{Buildpath}(z,y)
\]
$Mid(x, y)$ returns a midpoint (hop distance) on some min-cost $x$-$y$ path.

We now have a simple recursive procedure for building min-cost path

**Buildpath**(x,y)

If $y_d = x_{d+1}$

return $(x \to y)$

else

$z = Mid(x, y)$

Buildpath(x,z)

Buildpath(z,y)
Mid(x, y) returns a midpoint (hop distance) on some min-cost x-y path.

We now have a simple recursive procedure for building min-cost path:

**Buildpath(x,y)**

- If $y_d = x_{d+1}$
  - return $(x \rightarrow y)$
- else
  - $z = Mid(x, y)$
  - Buildpath(x,z)
  - Buildpath(z,y)
\textbf{Buildpath}(x,y)

If $y_d = x_{d+1}$
return $(x \rightarrow y)$
else
$z = \text{Mid}(x, y)$
Buildpath(x,z)
Buildpath(z,y)
**Buildpath**($x,y$)

If $y_d = x_{d+1}$
return $(x \rightarrow y)$
else
$z = Mid(x, y)$
Buildpath($x,z$)
Buildpath($z,y$)

**Lemma:** If $Mid(x, y)$ uses $O(D + n)$ space
⇒ Buildpath($0,F$) uses $O(D + n)$ space
**Buildpath(x,y)**

If $y_d = x_{d+1}$

return $(x \rightarrow y)$

else

$z = \text{Mid}(x, y)$

Buildpath(x,z)

Buildpath(z,y)

**Lemma:** If $\text{Mid}(x, y)$ uses $O(D + n)$ space

$\Rightarrow$ Buildpath(0,F) uses $O(D + n)$ space

**Lemma:** Let $\text{Area}(x, y)$ be area of $x, y$ box

If $\text{Mid}(x, y)$ uses $O(\text{Area}(x, y))$ time

$\Rightarrow$ Buildpath(0,F) uses $O(Dn)$ time
\textbf{Buildpath}(x,y)

\begin{align*}
\text{If } y_d &= x_{d+1} \\
&\quad \text{return } (x \rightarrow y) \\
\text{else} \\
&\quad z = \text{Mid}(x, y) \\
&\quad \text{Buildpath}(x,z) \\
&\quad \text{Buildpath}(z,y)
\end{align*}

\textbf{Lemma:} Let \textit{Area}(x, y) be area of \textit{x, y} box

If \textit{Mid}(x, y) \text{ uses } O(\textit{Area}(x, y)) \text{ time}

\implies \text{Buildpath}(0,F) \text{ uses } O(Dn) \text{ time}
**Buildpath(x,y)**

If $y_d = x_{d+1}$
   
   return $(x \rightarrow y)$

else
   
   $z = \text{Mid}(x, y)$
   
   Buildpath(x,z)
   
   Buildpath(z,y)

Lemma: Let $\text{Area}(x, y)$ be area of $x, y$ box

   If $\text{Mid}(x, y)$ uses $O(\text{Area}(x, y))$ time
   
   $\Rightarrow$ Buildpath(0,F) uses $O(Dn)$ time

Proof: Rectangles at recursion level $i$ are height $\leq D/2^i$

   $\Rightarrow$ Total work at level $i$ is $\leq nD/2^i$

   $\Rightarrow$ Total work $\leq$
**Buildpath(x,y)**

If \( y_d = x_{d+1} \)

return \((x \rightarrow y)\)

else

\( z = \text{Mid}(x, y) \)

Buildpath(x,z)

Buildpath(z,y)

---

**Lemma:** Let \( \text{Area}(x,y) \) be area of \( x, y \) box

If \( \text{Mid}(x, y) \) uses \( O(\text{Area}(x,y)) \) time

\( \Rightarrow \) Buildpath(0,F) uses \( O(Dn) \) time

**Proof:** Rectangles at recursion level \( i \) are height \( \leq D/2^i \)

\( \Rightarrow \) Total work at level \( i \) is \( \leq nD/2^i \)

\( \Rightarrow \) Total work \( \leq n \left( \frac{D}{2^0} \right) \)
**Buildpath(x,y)**
If $y_d = x_{d+1}$
   return $(x \rightarrow y)$
else
   $z = Mid(x, y)$
   Buildpath(x,z)
   Buildpath(z,y)

Lemma: Let $Area(x, y)$ be area of $x, y$ box

If $Mid(x, y)$ uses $O(Area(x, y))$ time
⇒ Buildpath(0,F) uses $O(Dn)$ time

Proof: Rectangles at recursion level $i$ are height $\leq D/2^i$
⇒ Total work at level $i$ is $\leq nD/2^i$
⇒ Total work $\leq n \left( \frac{D}{2^0} + \frac{D}{2^1} \right)$
Buildpath(x,y)

If \( y_d = x_{d+1} \)

return \((x \rightarrow y)\)

else

\[ z = \text{Mid}(x, y) \]

Buildpath(x,z)

Buildpath(z,y)

Lemma: Let \( \text{Area}(x, y) \) be area of \( x, y \) box

If \( \text{Mid}(x, y) \) uses \( O(\text{Area}(x, y)) \) time

\[ \Rightarrow \text{Buildpath}(0,F) \text{ uses } O(Dn) \text{ time} \]

Proof: Rectangles at recursion level \( i \) are height \( \leq \frac{D}{2^i} \)

\[ \Rightarrow \text{Total work at level } i \text{ is } \leq nD/2^i \]

\[ \Rightarrow \text{Total work } \leq n \left( \frac{D}{2^0} + \frac{D}{2^1} + \frac{D}{2^2} \right) \]
**Buildpath(x,y)**

If \( y_d = x_{d+1} \)

return \((x \rightarrow y)\)

else

\[ z = \text{Mid}(x, y) \]

Buildpath(x,z)

Buildpath(z,y)

---

**Lemma:** Let \( \text{Area}(x, y) \) be area of \( x, y \) box

If \( \text{Mid}(x, y) \) uses \( O(\text{Area}(x, y)) \) time

\[ \Rightarrow \text{Buildpath}(0,F) \text{ uses } O(Dn) \text{ time} \]

**Proof:** Rectangles at recursion level \( i \) are height \( \leq D/2^i \)

\[ \Rightarrow \text{Total work at level } i \text{ is } \leq nD/2^i \]

\[ \Rightarrow \text{Total work } \leq n \left( \frac{D}{2^0} + \frac{D}{2^1} + \frac{D}{2^2} + \frac{D}{2^3} + \cdots \right) \leq 2nD \]
Just saw that if $\text{Mid}(x, y)$ can be implemented using $O(D + n)$ space and $\text{Area}(x, y)$ time, then path can be built using $O(D + n)$ space and $O(Dn)$ time.

There are two different methods in literature for implementing $\text{Mid}(x, y)$. They can both be used here, but we will use (b).

(a) Hirschberg ('75)
For longest common subsequence problem.
Runs two modified Dijkstra’s that meet in “middle”
Every vertex had constant outdegree ($\leq 3$)
Used extensively in bioinformatics.

(b) Munro & Ramirez ('82)
For graphs like our’s
Runs one modified Dijkstra
Uses $\Theta(Dn^2)$ time (we can improve to $\Theta(Dn)$ with Monge)
Implementing $Mid(x, y)$ in $O(D + n)$ space and $Area(x, y)$ time

For every $z$, let $C(z)$ be min cost path distance from $x$ to $z$. For $z_d \geq \bar{d}$, let $P(z)$ be a point on level $\bar{d}$ lying on some min-cost path.
Implementing $\text{Mid}(x, y)$ in $O(D + n)$ space and $\text{Area}(x, y)$ time

For every $z$, let $C(z)$ be min cost path distance from $x$ to $z$.
For $z_d \geq \bar{d}$, let $P(z)$ be a point on level $\bar{d}$ lying on some min-cost path.

If $z_d = \bar{d}$, $P(z) = z$.
If $z_d > \bar{d}$, then $P(z) = P(\text{pred}(z))$
where $\text{pred}(z)$ is predecessor of $z$ on min cost path.
Implementing $\text{Mid}(x, y)$ in $O(D + n)$ space and $\text{Area}(x, y)$ time

For every $z$, let $C(z)$ be min cost path distance from $x$ to $z$. For $z_d \geq \bar{d}$, let $P(z)$ be a point on level $\bar{d}$ lying on some min-cost path.

If $z_d = \bar{d}$, $P(z) = z$.
If $z_d > \bar{d}$, then $P(z) = P(\text{pred}(z))$ where $\text{pred}(z)$ is predecessor of $z$ on min cost path.

All of the $C(z)$ and $P(z)$ on level $d$ can be calculated in $O(y_d - x_d)$ time (Monge property) using only knowledge of $C(z')$ and $P(z')$ where $z'$ on level $\bar{d} - 1$. 

\begin{align*}
\text{Implementing } & \text{Mid}(x, y) \text{ in } O(D + n) \text{ space and } \text{Area}(x, y) \text{ time} \\
\text{For every } z, \text{ let } C(z) \text{ be min cost path distance from } x \text{ to } z. \\
\text{For } z_d \geq \bar{d}, \text{ let } P(z) \text{ be a point on level } \bar{d} \text{ lying on some min-cost path.} \\
\text{If } z_d = \bar{d}, \text{ } P(z) = z. \\
\text{If } z_d > \bar{d}, \text{ then } P(z) = P(\text{pred}(z)) \text{ where } \text{pred}(z) \text{ is predecessor of } z \text{ on min cost path.} \\
\text{All of the } C(z) \text{ and } P(z) \text{ on level } d \text{ can be calculated in } O(y_d - x_d) \text{ time (Monge property) using only knowledge of } C(z') \text{ and } P(z') \text{ where } z' \text{ on level } \bar{d} - 1. 
\end{align*}
Implementing $\text{Mid}(x, y)$ in $O(D + n)$ space and $\text{Area}(x, y)$ time

For every $z$, let $C(z)$ be min cost path distance from $x$ to $z$.
For $z_d \geq \bar{d}$, let $P(z)$ be a point on level $\bar{d}$ lying on some min-cost path.

If $z_d = \bar{d}$, $P(z) = z$.
If $z_d > \bar{d}$, then $P(z) = P(\text{pred}(z))$ where $\text{pred}(z)$ is predecessor of $z$ on min cost path.

All of the $C(z)$ and $P(z)$ on level $d$ can be calculated in $O(y_d - x_d)$ time (Monge property) using only knowledge of $C(z')$ and $P(z')$ where $z'$ on level $d - 1$. 
Implementing $\text{Mid}(x, y)$ in $O(D + n)$ space and $\text{Area}(x, y)$ time

For every $z$, let $C(z)$ be min cost path distance from $x$ to $z$.

For $z_d \geq \bar{d}$, let $P(z)$ be a point on level $\bar{d}$ lying on some min-cost path.

If $z_d = \bar{d}$, $P(z) = z$.
If $z_d > \bar{d}$, then $P(z) = P(\text{pred}(z))$
where $\text{pred}(z)$ is predecessor of $z$ on min cost path.

All of the $C(z)$ and $P(z)$ on level $d$ can be calculated in $O(y_d - x_d)$ time (Monge property) using only knowledge of $C(z')$ and $P(z')$ where $z'$ on level $d - 1$. 
Implementing $\text{Mid}(x, y)$ in $O(D + n)$ space and $\text{Area}(x, y)$ time

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$\Rightarrow$ $Buildpath((0, 0), (n, D))$ uses $O(D + n)$ space and $O(Dn)$ time
Implemented $\text{Mid}(x, y)$ in $O(D + n)$ space and $\text{Area}(x, y)$ time

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$$H(i, d) = \min_{0 \leq j < i} \left( H(j, d - 1) + w(j, i) \right) \quad 0 \leq d \leq D \leq n$$
Implemented \( \text{Mid}(x, y) \) in \( O(D + n) \) space and \( \text{Area}(x, y) \) time

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\[ H(i, d) = \min_{0 \leq j < i} \left( H(j, d - 1) + w(j, i) \right) \quad 0 \leq i \leq n, 0 \leq d \leq D \]

\[ \text{using } O(D + n) \text{ space and } O(Dn) \text{ time} \]
Outline

• Review of the Monge Speedup

• Saving Space While Saving Time

• Conclusion
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We just saw one technique for reducing time in dynamic programming and another for reducing space.

There are *many* such DP improvement techniques.

The problem is that they’re all ad-hoc techniques, primarily known to specialists.

Need to develop a general theory of DP improvements, especially speedups, that is accessible to “users”.

Goal is a recipe book that DP designers can check to see how to speed up their application-specific problems.
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Thank You. Questions?
Open Question

- Two-Sided Online K-Median on a Line

Identify $k$ nodes as service centers. Cost of servicing request $w_i$, is $w_i$ times distance from node $i$ to nearest service center. Problem is to find location of $k$ service centers that minimize total service cost.
Two-Sided Online K-Median on a Line

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- Naive DP: \( O(kn^2) \)
- Using Monge property: \( O(kn) \)
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Online Problem: Adding new elements to right and left. Best known is $O(kn)$. Just as bad as reconstructing from scratch. Is there a better way?