Perspective

Hamming weights in irreducible cyclic codes✩

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ABSTRACT

The objectives of this paper are to survey and extend earlier results on the weight distributions of irreducible cyclic codes, present a divisibility theorem and develop bounds on the weights in irreducible cyclic codes.

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1. Introduction

Irreducible cyclic codes are an interesting type of codes and have applications in space communications. They have been studied for decades and a lot of progress has been made.

Throughout this paper, let p be a prime, q = p s for a positive integer s, and r = q m for a positive integer m. A linear [n, k, d] code over GF(q) is a k-dimensional subspace of GF(q) n with minimum (Hamming) distance d. Let A i denote the number of codewords with Hamming weight i in a code C of length n. The weight enumerator of C is defined by

\[ 1 + A_1 x + A_2 x^2 + \cdots + A_n x^n. \]

A linear [n, k] code C over the finite field GF(q) is called cyclic if \((c_0, c_1, \ldots, c_{n-1}) \in C\) implies \((c_{n-1}, c_0, c_1, \ldots, c_{n-2}) \in C\). Let \(\gcd(n, q) = 1\). By identifying any vector \((c_0, c_1, \ldots, c_{n-1}) \in GF(q)^n\) with \(c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1} x^{n-1} \in GF(q)[x]/(x^n - 1)\), any code C of length n over GF(q) corresponds to a subset of GF(q)[x]/(x^n - 1). The linear code C is cyclic if and only if the corresponding subset in GF(q)[x]/(x^n - 1) is an ideal of the ring GF(q)[x]/(x^n - 1).

Note that every ideal of GF(q)[x]/(x^n - 1) is principal. Let \(C = (g(x))\) be a cyclic code, where g(x) is monic and deg(f) is minimal. Then g(x) is called the generator polynomial and h(x) = (x^n - 1)/g(x) is referred to as the parity-check polynomial of C.

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Let $N > 1$ be an integer dividing $r - 1$, and put $n = (r - 1)/N$. Let $\alpha$ be a primitive element of $\text{GF}(r)$ and let $\theta = \alpha^N$. The set
\[ C(r, N) = \{ (\text{Tr}_{r/q}(\beta), \text{Tr}_{r/q}(\beta\theta), \ldots, \text{Tr}_{r/q}(\beta\theta^{n-1})) : \beta \in \text{GF}(r) \} \]
is called an irreducible cyclic $[n, m_0]_q$ code over $\text{GF}(q)$, where $\text{Tr}_{r/q}$ is the trace function from $\text{GF}(r)$ onto $\text{GF}(q)$, $m_0$ is the multiplicative order of $q$ modulo $n$ and $m_0$ divides $n$.

Irreducible cyclic codes have been an interesting subject of study for many years. The celebrated Golay code is an irreducible cyclic code and was used on the Mariner Jupiter–Saturn Mission. They form a special class of codes and are interesting in theory as they are minimal cyclic codes. The weight distribution, i.e., the vector $(1, A_1, A_2, \ldots, A_{n-1})$, of the irreducible cyclic codes has been determined for a small number of special cases.

The objectives of this paper are to survey and extend earlier results on the weight distributions of irreducible cyclic codes (see Theorems 23, 21, 15, 17, 18, 19 and 20 as extensions and generalizations of earlier results), to completely characterize one-weight irreducible cyclic codes (Theorem 16), which is an extension of the result in [28], and to present a divisibility theorem and develop bounds on the weights in irreducible cyclic codes (see Theorems 24 and 13).

2. Group characters, cyclotomy, and Gaussian periods

In this section, we present results on group characters, cyclotomy and Gaussian sums which will be needed in the sequel.

2.1. Group characters and Gaussian sums

Let $\text{Tr}_{q/p}$ denote the trace function from $\text{GF}(q)$ to $\text{GF}(p)$. An additive character of $\text{GF}(q)$ is a nonzero function $\chi$ from $\text{GF}(q)$ to the set of complex numbers such that $\chi(x + y) = \chi(x)\chi(y)$ for any pair $(x, y) \in \text{GF}(q)^2$. For each $b \in \text{GF}(q)$, the function
\[ \chi_b(c) = e^{2\pi \sqrt{-\text{Tr}_{q/p}(bc)/p}} \quad \text{for all } c \in \text{GF}(q) \]
defines an additive character of $\text{GF}(q)$. When $b = 0$, $\chi_b(c) = 1$ for all $c \in \text{GF}(q)$, and is called the trivial additive character of $\text{GF}(q)$. The character $\chi_1$ in (2) is called the canonical additive character of $\text{GF}(q)$.

A multiplicative character of $\text{GF}(q)$ is a nonzero function $\psi$ from $\text{GF}(q)^*$ to the set of complex numbers such that $\psi(xy) = \psi(x)\psi(y)$ for all pairs $(x, y) \in \text{GF}(q)^* \times \text{GF}(q)^*$. Let $g$ be a fixed primitive element of $\text{GF}(q)$. For each $j = 1, 2, \ldots, q - 1$, the function $\psi_j$ with
\[ \psi_j(g^k) = e^{2\pi \sqrt{-\text{Tr}_q(k/q-1)}} \quad \text{for } k = 0, 1, \ldots, q - 2 \]
defines a multiplicative character with order $(q - 1)/\text{gcd}(q - 1, j)$ of $\text{GF}(q)$. When $j = q - 1$, $\psi_0(c) = 1$ for all $c \in \text{GF}(q)^*$, and is called the trivial multiplicative character of $\text{GF}(q)$.

Let $q$ be odd and $j = (q - 1)/2$ in (3), we then get a multiplicative character $\eta$ such that $\eta(c) = 1$ if $c$ is the square of an element and $\eta(c) = -1$ otherwise. This $\eta$ is called the quadratic character of $\text{GF}(q)$.

Let $\psi$ be a multiplicative character with order $k$ where $k|(q - 1)$ and $\chi$ an additive character of $\text{GF}(q)$. Then the Gaussian sum $G(\psi, \chi)$ of order $k$ is defined by
\[ G(\psi, \chi) = \sum_{c \in \text{GF}(q)^*} \psi(c)\chi(c). \]

Since $G(\psi, \chi_b) = \tilde{\psi}(b)G(\psi, \chi_1)$, we just consider $G(\psi, \chi_1)$, briefly denoted as $G(\psi)$, in the sequel. If $\psi \neq \psi_0$, then
\[ |G(\psi)| = q^{1/2}. \]

Generally, to explicitly determine the value of Gaussian sums is a challenging task. At present, they can be determined in a few cases. Among them is the following case of $k = 2$.

If $q = p^s$, where $p$ is an odd prime and $s$ is a positive integer, then
\[ G(\eta) = \begin{cases} (-1)^{s-1}q^{1/2} & \text{if } p \equiv 1 \pmod{4}, \\ (-1)^{s-1}(\sqrt{-1})^s q^{1/2} & \text{if } p \equiv 3 \pmod{4}. \end{cases} \]

The following result [15] is useful in the sequel.

**Lemma 1.** Let $\chi$ be a nontrivial additive character of $\text{GF}(q)$ with $q$ odd, and let $f(x) = a_2x^2 + a_1x + a_0 \in \text{GF}(q)[x]$ with $a_2 \neq 0$. Then
\[ \sum_{c \in \text{GF}(q)} \chi(f(c)) = \chi(a_0 - a_2^{-1}(4a_2)^{-1})\eta(a_2)G(\eta). \]
The Gaussian sums of small order, such as $k = 3, 4, 5, 6,$ and $12,$ can be also determined; see [5]. In another special case, called “semi-primitive” case, the Gaussian sums are known and given in the following lemma [5].

**Lemma 2.** Assume that $N \neq 2$ and there exists a positive integer $j$ such that $p^j \equiv -1 \pmod{N},$ and the $j$ is the least such. Let $q = p^{2j}$ for some integer $\gamma.$ Then the Gaussian sums of order $N$ over $\text{GF}(q)$ are given by

$$G(\gamma) = \begin{cases} (-1)^{N-1} \sqrt{q}, & \text{if } p = 2, \\ (-1)^{N-1} \frac{q^{\frac{p^j+1}{2}} - 1}{\sqrt{q}}, & \text{if } p > 3. \end{cases}$$

Furthermore, for $1 \leq i \leq N - 1,$ the Gaussian sums $G(\gamma^i)$ are given by

$$G(\gamma^i) = \begin{cases} (-1)^i \sqrt{q}, & \text{if } N \text{ is even, } p, \gamma \text{ and } \frac{p^j + 1}{N} \text{ are odd}; \\ (-1)^{N-1} \sqrt{q}, & \text{otherwise}. \end{cases}$$

If $p$ generates a subgroup of the group $\left(\mathbb{Z}/N\mathbb{Z}\right)^*$ with index $\left([\mathbb{Z}/N\mathbb{Z}]^* : \langle p \rangle \right] = 2$ and $-1 \notin \langle p \rangle \subset \left(\mathbb{Z}/N\mathbb{Z}\right)^*,$ which is the so-called “quadratic residues” or “index 2” case, Gaussian sums are also explicitly determined. See [31] and its references for details. We list one of the results [31] in the index 2 case below, which is useful in the sequel.

**Lemma 3.** Let $N_1 = p^k$ where $3 \neq l \equiv 3 \pmod{4}$ is a prime and $\lambda$ is a positive integer. Let $f = \text{ord}_{N_1}(p), \ r = p^b$ for some positive integer $s,$ and $\psi$ be a primitive multiplicative character of order $N_1$ over $\text{GF}(r)^*.$ Assume that $f = \frac{q(N_1)}{2},$ which means that $p$ generates the quadratic residues modulo $N_1,$ then, for $1 \leq t \leq \lambda,$ we have that

$$G(\psi^{t-1}) = (-1)^{t-1} \cdot p^\frac{qf(bh^{t-1})}{2} \cdot \left(\frac{a + b\sqrt{-l}}{2}\right)^{sp^{t-1}}$$

$$:= p^{(s,\lambda)}_t \left(A^{(s,\lambda)}_t + B^{(s,\lambda)}_t \sqrt{-l}\right),$$

where $h$ is the ideal class number of $\mathbb{Q}(\sqrt{-l}), a, b$ are integers given by

$$\begin{cases} a^2 + lb^2 = 4p^h \\ a \equiv -2p^{\frac{l-1+2h}{4}} \pmod{l}, \end{cases}$$

and $p^{(s,\lambda)}_t, A^{(s,\lambda)}_t, B^{(s,\lambda)}_t \in \mathbb{Z}$ are defined as

$$p^{(s,\lambda)}_t = (-1)^{t-1} \cdot p^\frac{qf(bh^{t-1})}{2};$$

$$A^{(s,\lambda)}_t = \text{Re} \left(\frac{a + b\sqrt{-l}}{2}\right)^{sp^{t-1}}; \quad B^{(s,\lambda)}_t = \text{Im} \left(\frac{a + b\sqrt{-l}}{2}\right)^{sp^{t-1}} \left/ \sqrt{i}. \right.$$

(7)

2.2. Cyclotomy

Let $r - 1 = nN$ for two positive integers $n > 1$ and $N > 1,$ and let $\alpha$ be a fixed primitive element of $\text{GF}(r).$ Define $c^{(N,r)}_i = \alpha^{(i)(\alpha^N)}$ for $i = 0, 1, \ldots, N - 1,$ where $(\alpha^N)$ denotes the subgroup of $\text{GF}(r)^*$ generated by $\alpha^N.$ The cosets $c^{(N,r)}_i$ are called the **cyclotomic classes of order $N$ in $\text{GF}(r).$** The **cyclotomic numbers** of order $N$ are defined by

$$(i, j)^{(N,r)} = \left|(c^{(N,r)}_i + 1) \cap c^{(N,r)}_j\right|$$

for all $0 \leq i \leq N - 1$ and $0 \leq j \leq N - 1.$

We will need the following lemma [10] in the sequel.

**Lemma 4.** Let $r - 1 = nN$ and let $q$ be a prime power. Then

$$\sum_{u=0}^{N-1} (u, u + k)^{(N,r)} = \begin{cases} n - 1, & \text{if } k = 0; \\ n, & \text{if } k \neq 0. \end{cases}$$

To determine the weight distribution of some classes of linear codes in the sequel, we need the following lemma.
Lemma 5. Let $e_1$ be a positive divisor of $r - 1$ and let $i$ be any integer with $0 \leq i < e_1$. We have the following multiset equality:

$$\left\{ xy : y \in \text{GF}(q)^*, \ x \in C_i^{(e_1,r)} \right\} = \frac{(q - 1) \gcd((r - 1)/(q - 1), e_1)}{e_1} \ast C_i^{(\gcd((r - 1)/(q - 1), e_1), r)},$$

where $\frac{(q - 1) \gcd((r - 1)/(q - 1), e_1)}{e_1} \ast C_i^{(\gcd((r - 1)/(q - 1), e_1), r)}$ denotes the multiset in which each element in the set $C_i^{(\gcd((r - 1)/(q - 1), e_1), r)}$ appears in the multiset with multiplicity $\frac{(q - 1) \gcd((r - 1)/(q - 1), e_1)}{e_1}$.

Proof. We just prove the conclusion for $i = 0$. The proof is similar for $i \neq 0$ since

$$C_i^{(\gcd((r - 1)/(q - 1), e_1), r)} = \alpha C_0^{(\gcd((r - 1)/(q - 1), e_1), r)}.$$

Note that every $y \in \text{GF}(q)^*$ can be expressed as $y = \alpha^{\frac{r - 1}{\ell}}$ for an unique $\ell$ with $0 \leq \ell < q - 1$ and every $x \in C_0^{(e_1,r)}$ can be expressed as $x = \alpha^{e_1 j}$ for an unique $j$ with $0 \leq j < (r - 1)/e_1$. Then we have

$$xy = \alpha^{\frac{r - 1}{\ell} + e_1 j}.$$

It follows that

$$xy = \alpha^{\frac{r - 1}{\ell} - e_1 j} = \left(\alpha^{\gcd((r - 1)/(q - 1), e_1)}\right)^{\frac{r - 1}{\gcd((r - 1)/(q - 1), e_1)}} \ast \left(\alpha^{\gcd((r - 1)/(q - 1), e_1)}\right)^{\frac{e_1}{\gcd((r - 1)/(q - 1), e_1)}}.$$

Note that

$$\gcd\left(\frac{r - 1}{\gcd((r - 1)/(q - 1), e_1)}, \frac{e_1}{\gcd((r - 1)/(q - 1), e_1)}\right) = 1.$$

When $\ell$ ranges over $0 \leq \ell < q - 1$ and $j$ ranges over $0 \leq j < (r - 1)/e_1$, $xy$ takes on the value 1 exactly $\frac{q - 1}{e_1} \gcd((r - 1)/(q - 1), e_1)$ times.

Let $x_1 \in C_0^{(e_1,r)}$ for $i_1 = 1$ and $i_1 = 2$, and let $y_1 \in \text{GF}(q)^*$ for $i_2 = 1$ and $i_2 = 2$. Then $\frac{x_1}{x_2} \in C_0^{(e_1,r)}$ and $\frac{y_1}{y_2} \in \text{GF}(q)^*$. Note that $x_1 y_1 = x_2 y_2$ if and only if $\frac{x_1}{x_2} = \frac{y_1}{y_2} = 1$. Then the conclusion of the lemma for the case $i = 0$ follows from the discussions above.

2.3. Gaussian periods

The Gaussian periods are defined by

$$\eta_i^{(N,r)} = \sum_{x \in C_i^{(N,r)}} \chi(x), \quad i = 0, 1, \ldots, N - 1,$$

where $\chi$ is the canonical additive character of $\text{GF}(r)$.

The following lemma presents some basic properties of Gaussian periods, and will be useful later.

Lemma 6 ([26]). Let symbols be the same as before. Then we have

1. $\sum_{i=0}^{N-1} \eta_i = -1$.
2. $\sum_{i=0}^{N-1} \eta_i \eta_{i+k} = r \theta_k - n$ for all $k \in \{0, 1, \ldots, N - 1\}$, where

$$\theta_k = \begin{cases} 1 & \text{if } n \text{ is even and } k = 0 \\ 1 & \text{if } n \text{ is odd and } k = N/2 \\ 0 & \text{otherwise}, \end{cases}$$

and equivalently $\theta_k = 1$ if and only if $-1 \in C_k^{(N,r)}$.

Gaussian periods are closely related to Gaussian sums. By the discrete Fourier transform, it is known that

$$\eta_i^{(N,r)} = \frac{1}{N} \sum_{j=0}^{N-1} \zeta_N^{-ij} G(\psi^j) = \frac{1}{N} \left[ -1 + \sum_{j=1}^{N-1} \zeta_N^{-ij} G(\psi^j) \right].$$

(9)

where $\zeta_N = e^{2\pi i/N}$ and $\psi$ is a primitive multiplicative character of order $N$ over $\text{GF}(r)^*$.

The values of the Gaussian periods in general are also very hard to compute. However, they can be computed in a few cases. To present some known results on Gaussian periods, we need to introduce period polynomials.
The period polynomials $\psi_{(N,r)}(X)$ are defined by

$$
\psi_{(N,r)}(X) = \prod_{i=0}^{N-1} (X - \eta_i^{(N,r)}).
$$

It is known that $\psi_{(N,r)}(X)$ is a polynomial with integer coefficients [21]. We will need the following four lemmas whose proofs can be found in [21].

**Lemma 7.** Let $N = 3$. Let $c$ and $d$ be defined by $4r = c^2 + 27d^2$, $c \equiv 1 \pmod 3$, and, if $p \equiv 1 \pmod 3$, then $\gcd(c, p) = 1$. These restrictions determine $c$ uniquely, and $d$ up to sign. Then we have

$$
\psi_{(3,r)}(X) = X^3 + X^2 - \frac{r-1}{3} X - \frac{(c+3)r - 1}{27}.
$$

**Lemma 8.** Let $N = 3$. We have the following results on the factorization of $\psi_{(3,r)}(X)$.

(a) If $p \equiv 2 \pmod 3$, then $s \cdot m$ is even, and

$$
\psi_{(3,r)}(X) = \begin{cases} 
3^{-3}(3X + 1 + 2\sqrt{7})(3X + 1 - \sqrt{7})^2 & \text{if } s \cdot m/2 \text{ even}, \\
3^{-3}(3X + 1 - 2\sqrt{7})(3X + 1 + \sqrt{7})^2 & \text{if } s \cdot m/2 \text{ odd}.
\end{cases}
$$

(b) If $p \equiv 1 \pmod 3$, and $s \cdot m \not\equiv 0 \pmod 3$, then $\psi_{(3,r)}(X)$ is irreducible over the rationals.

(c) If $p \equiv 1 \pmod 3$, and $s \cdot m \equiv 0 \pmod 3$, then

$$
\psi_{(3,r)}(X) = \frac{1}{27} \left(3X + 1 - c_1 r \frac{1}{2} \right) \left(3X + 1 + \frac{1}{2} (c_1 + 9d_1) r \frac{1}{2} \right) \cdot \left(3X + 1 + \frac{1}{2} (c_1 - 9d_1) r \frac{1}{2} \right),
$$

where $c_1$ and $d_1$ are given by $4p^{s/m/3} = c_1^2 + 27d_1^2$, $c_1 \equiv 1 \pmod 3$ and $\gcd(c_1, p) = 1$.

**Lemma 9.** Let $N = 4$. Let $u$ and $v$ be defined by $r = u^2 + 4v^2$, $u \equiv 1 \pmod 4$, and, if $p \equiv 1 \pmod 4$, then $\gcd(u, p) = 1$. These restrictions determine $u$ uniquely, and $v$ up to sign.

If $n$ is even, then

$$
\psi_{(4,r)}(X) = X^4 + X^2 - \frac{3r-3}{8} X^2 + \frac{(2u-3)r+1}{16} X + \frac{r^2 - (4u^2 - 8u + 6)r + 1}{256}.
$$

If $n$ is odd, then

$$
\psi_{(4,r)}(X) = X^4 + X^2 + \frac{r+3}{8} X^2 + \frac{(2u+1)r+1}{16} X + \frac{9r^2 - (4u^2 - 8u - 2)r + 1}{256}.
$$

**Lemma 10.** Let $N = 4$. We have the following results on the factorization of $\psi_{(4,r)}(X)$.

(a) If $p \equiv 3 \pmod 4$, then $s \cdot m$ is even, and

$$
\psi_{(4,r)}(X) = \begin{cases} 
4^{-3}(4X + 1 + 3\sqrt{7})(4X + 1 - \sqrt{7})^3 & \text{if } s \cdot m/2 \text{ even}, \\
4^{-3}(4X + 1 - 3\sqrt{7})(4X + 1 + \sqrt{7})^3 & \text{if } s \cdot m/2 \text{ odd}.
\end{cases}
$$

(b) If $p \equiv 1 \pmod 4$, and $s \cdot m$ is odd, then $\psi_{(4,r)}(X)$ is irreducible over the rationals.

(c) If $p \equiv 1 \pmod 4$, and $s \cdot m \equiv 2 \pmod 4$, then

$$
\psi_{(4,r)}(X) = 4^{-4} \left((4X + 1)^2 + 2\sqrt{r}(4X + 1) - r - 2\sqrt{r}u \right) \left((4X + 1)^2 - 2\sqrt{r}(4X + 1) - r + 2\sqrt{r}u \right),
$$

the quadratics being irreducible. The $u$ is defined in Lemma 9.

(d) If $p \equiv 1 \pmod 4$, and, $s \cdot m \equiv 0 \pmod 4$, then

$$
\psi_{(4,r)}(X) = 4^{-4} \left((4X + 1) + \sqrt{r} + 2r^{1/4}u_1 \right) \left((4X + 1) + \sqrt{r} - 2r^{1/4}u_1 \right) \times \left((4X + 1) - \sqrt{r} + 4r^{1/4}v_1 \right) \left((4X + 1) - \sqrt{r} - 4r^{1/4}v_1 \right),
$$

where $u_1$ and $v_1$ are given by $p^{s/m/2} = u_1^2 + 4v_1^2$, $u_1 \equiv 1 \pmod 4$ and $\gcd(u_1, p) = 1$.

The following lemma follows from Lemma 1 and (5).

**Lemma 11.** When $N = 2$, the Gaussian periods are given by the following:

$$
\eta_0^{(2,r)} = \begin{cases} 
-1 + (-1)^{s-1} r^{1/2} & \text{if } p \equiv 1 \pmod 4 \\
-1 + \frac{2}{2} (-1)^{s-1} (-1)^{s-m} r^{1/2} & \text{if } p \equiv 3 \pmod 4
\end{cases}
$$

and $\eta_1^{(2,r)} = -1 - \eta_0^{(2,r)}$. 

By Lemma 2 and (9), the Gaussian periods in the semi-primitive case are known and are described in the following lemma [2,21].

Lemma 12. Assume that \( N > 2 \) and there exists a positive integer \( j \) such that \( p^j \equiv -1 \pmod{N} \), and the \( j \) is the least such. Let \( r = p^{j\gamma} \) for some integer \( \gamma \).

(a) If \( \gamma, p \) and \((p^j + 1)/N\) are all odd, then

\[
\eta_{n/2}^{(N,r)} = \frac{(N - 1)\sqrt{r} - 1}{N}, \quad \eta_k^{(N,r)} = -\frac{\sqrt{r} + 1}{N} \quad \text{for } k \neq N/2.
\]

(b) In all other cases,

\[
\eta_0^{(N,r)} = \frac{(-1)^{r+1}(N - 1)\sqrt{r} - 1}{N}, \quad \eta_k^{(N,r)} = \frac{(-1)^r \sqrt{r} - 1}{N} \quad \text{for } k \neq 0.
\]

From Lemma 3 and (9), the Gaussian periods in the so-called quadratic residues (or index 2) case can be also computed. The results with \( 3 \neq N \equiv 3 \pmod{4} \) being odd prime are given in [4,21].

3. The weights in irreducible cyclic codes

Let \( N > 1 \) be an integer dividing \( r - 1 \), and put \( n = (r - 1)/N \). Let \( \alpha \) be a primitive element of \( \text{GF}(r) \) and let \( \theta = \alpha^N \). Let \( Z(r, a) \) denote the number of solutions \( x \in \text{GF}(r) \) of the equation \( \text{Tr}_{\gamma/q}(\alpha x^N) = 0 \). Let \( \zeta_p = e^{2\pi \sqrt{-1}/p} \), and \( \chi(x) = \zeta_p^{\text{Tr}_{r/p}(x)} \), where \( \text{Tr}_{r/p} \) is the trace function from \( \text{GF}(r) \) to \( \text{GF}(p) \). Then \( \chi \) is an additive character of \( \text{GF}(r) \). We have then by Lemma 5

\[
Z(r, a) = \frac{1}{q} \sum_{\gamma \in \text{GF}(q)} \sum_{x \in \text{GF}(r)} \zeta_p^{\text{Tr}_{r/p}(\gamma \alpha x^N)}
\]

\[
= \frac{1}{q} \sum_{y \in \text{GF}(q)} \sum_{x \in \text{GF}(r)} \chi(\gamma ax^N)
\]

\[
= \frac{1}{q} \left[ q + r - 1 + N \sum_{y \in \text{GF}(q)^*} \sum_{x \in \mathbb{C}_d^{(N,r)}} \chi(\gamma ax) \right]
\]

\[
= \frac{1}{q} \left[ q + r - 1 + (q - 1) \gcd\left(\frac{r - 1}{q - 1}, N\right) \sum_{z \in \mathbb{C}_d^{(\text{gcd}\left(\frac{r - 1}{q - 1}, N\right), r)}} \chi(az) \right].
\]

Then the Hamming weight of the codeword

\( (\text{Tr}_{r/q}(\beta), \text{Tr}_{r/q}(\beta \theta), \ldots, \text{Tr}_{r/q}(\beta \theta^{n-1})) \)

in the irreducible cyclic code of (1), for \( 0 \neq \beta \in \mathbb{C}_d^{(\text{gcd}\left(\frac{r - 1}{q - 1}, N\right), r)} \), is equal to

\[
\frac{n}{qN} Z(r, \beta) - 1 = \frac{(q - 1) \left( r - 1 - \gcd\left(\frac{r - 1}{q - 1}, N\right) \eta_1^{\text{gcd}\left(\frac{r - 1}{q - 1}, N\right), r} \right)}{qN}.
\]

The weight expression of (12) is the key observation of this paper and proves that the determination of the weight distribution of an irreducible cyclic code is equivalent to that of the Gaussian periods of order \( N_1 = \gcd((r - 1)/(q - 1), N) \). McEliece [17] gave a different proof of (12) by Gaussian sums. From (9) we know that the weights of an irreducible cyclic code can be expressed as a linear combination of Gaussian sums.

Theorem 13. Let \( N_1 = \gcd((r - 1)/(q - 1), N) \). Then for all \( i \) with \( 0 \leq i \leq N_1 - 1 \), we have

(i) \( \eta_i^{(N_1, r)} \in \mathbb{Z} \);

(ii) \( N_1 \eta_i^{(N_1, r)} + 1 \equiv 0 \pmod{q} \); and

(iii) \( \left| \eta_i^{(N_1, r)} + \frac{1}{N_1} \right| \leq \left\lfloor \frac{(N_1 - 1)\sqrt{T}}{N_1} \right\rfloor \).
Proof. The conclusions of Parts (i) and (ii) follow from (12) directly, and that of Part (iii) follows from (4) and (9). □

Theorem 13 is an interesting result in the theory of cyclotomy.

Theorem 14. Let \( N_1 = \gcd((r - 1)/(q - 1), N) \). Then the Hamming weight of every codeword in the irreducible cyclic code of (1) is divisible by

\[
\frac{(q - 1)}{\gcd(q - 1, N/N_1)}.
\]

Proof. By (12), the Hamming weight of every nonzero codeword is equal to

\[
\frac{q - 1 - r - (1 + N_1N)}{\gcd(q - 1, N/N_1)} q^{\frac{N}{\gcd(q - 1, N/N_1)}}.
\]

The desired conclusion then follows from the fact that

\[
\gcd\left(q - 1, q^{\frac{N}{\gcd(q - 1, N/N_1)}}\right) = 1. \quad \square
\]

Particularly, when \( N \) divides \((r - 1)/(q - 1)\), the Hamming weight of every codeword in the irreducible cyclic code of (1) is divisible by \( q - 1 \).

Example 1. Let \( q = 5 \), \( m = 4 \), \( N = 4 \). Then the irreducible cyclic code of (1) over GF(q) has length 156, dimension 4, and the following weight distribution:

\[
1 + 156x^{112} + 156x^{124} + 156x^{128} + 156x^{136}.
\]

So by Theorem 14, 4 is a common divisor of all nonzero weights. Note that

\[
\gcd(112, 124, 128, 136) = 4.
\]

Example 2. Let \( q = 3 \), \( m = 4 \), \( N = 2 \). Then the irreducible cyclic code of (1) over GF(q) has length 40, dimension 4, and the following weight distribution:

\[
1 + 40x^{24} + 40x^{30}.
\]

So by Theorem 14, 2 is a common divisor of all nonzero weights. Note that \( \gcd(24, 30) = 6 \).

4. The weight distribution when \( N_1 = \gcd((r - 1)/(q - 1), N) = 1 \)

Theorem 15. Let \( N \) be a positive divisor of \( r - 1 \) such that \( N_1 = \gcd((r - 1)/(q - 1), N) = 1 \). Then the set \( C(r, N) \) in (1) is a \([(q^m - 1)/N, m, (q - 1)q^{m-1}/N] \) constant-weight code with the weight enumerator

\[
1 + (r - 1)x^{\frac{(q-1)q^{m-1}}{N}}.
\]

Proof. Since \( N \) divides \( r - 1 \) and \( \gcd((r - 1)/(q - 1), N) = 1 \), \( N \) must divide \( q - 1 \). It follows that

\[
\gcd((r - 1)/(q - 1), N) = \gcd(m, N) = 1.
\]

Let \( \alpha \) be the generator of GF(r)*. For any \( a \neq 0 \), it follows from (12) and Lemma 11 that for any \( \beta \in GF(r)^* \) the Hamming weight of any codeword

\[
c(\beta) = (\text{Tr}_{r/q}(\beta), \text{Tr}_{r/q}(\beta\theta), \ldots, \text{Tr}_{r/q}(\beta\theta^{n-1}))
\]

of the code \( C(r, N) \) is equal to

\[
n - \frac{Z(r, \beta) - 1}{N} = \frac{(q - 1)q^{m-1}}{N}.
\]

The weight distribution and dimension of the code follow. This completes the proof. □

Theorem 16. Let \( N \) be a positive divisor of \( r - 1 \). Then the set \( C(r, N) \) in (1) is a \([(q^m - 1)/N, m] \) constant-weight code if and only if \( N_1 = \gcd((r - 1)/(q - 1), N) = 1 \).
Proof. Theorem 15 shows that the condition is sufficient. We now prove the necessity of the condition. Let \( N_1 = \gcd((r - 1)/(q - 1), N) \) and \( n_1 = (r - 1)/N_1 \). Assume that \( C(r, N) \) is a constant weight code. It then follows from (12) that \( 1 + N_1\eta_i \) is a constant \( \lambda \) for all \( i \). Define \( \zeta_i = 1 + N_1\eta_i \). Then the formulas in Lemma 6 becomes

1. \[ \sum_{i=0}^{N_1-1} \zeta_i = 0. \]
2. \[ \sum_{i=0}^{N_1-1} \zeta_i \zeta_{i+k} = N_1(N_1\theta_k - 1)r \quad \text{for all } k \in \{0, 1, \ldots, N_1 - 1\}, \]

where \( \theta_k = \begin{cases} 1 & \text{if } n_1 \text{ is even and } k = 0 \\ 1 & \text{if } n_1 \text{ is odd and } k = N_1/2 \\ 0 & \text{otherwise,} \end{cases} \)

and equivalently \( \theta_k = 1 \) if and only if \( -1 \in C_{0}^{(N_1, r)} \).

Since \( N_1 \) is a divisor of \( (r - 1)/(q - 1) \), \( \mathbb{GF}(q)^* \subset C_{0}^{(N_1, r)} \). It follows that \( \theta_0 = 1 \). Hence, we have \( N_1\lambda = 0, \quad N_1\lambda^2 = N_1(N_1 - 1)r \).

Whence, \( N_1 = 1 \). This completes the proof. \( \square \)

Theorem 16 above is a complete characterization of one-weight irreducible cyclic codes in the general case that \( N \) is any divisor of \( r - 1 \), which is different from Theorem 1 in [28], where Vega and Wolfmann considered only the case that \( N \) is a divisor of \( q - 1 \) and use the period of the parity-check polynomial of the code for the characterization. Theorem 15 is extension of Theorem 6 in [8].

5. The weight distribution when \( N_1 = \gcd((r - 1)/(q - 1), N) = 2 \)

Theorem 17. Let \( N \) be a positive divisor of \( r - 1 \). If \( N_1 = \gcd((r - 1)/(q - 1), N) = 2 \), then the set \( C(r, N) \) in (1) is a \( [(q^m - 1)/N, m, (q - 1)/(q - \sqrt{T})/Nq] \) two-weight code with the weight enumerator

\[ 1 + \frac{r - 1}{2} x \frac{(q - 1) \sqrt{T}}{q^m} + \frac{r - 1}{2} x \frac{(q - 1) \sqrt{T}}{q^m} + \frac{1}{3} x \frac{(q - 1) \sqrt{T}}{q^m} \]

Proof. Since \( N_1 = \gcd((r - 1)/(q - 1), N) = 2 \), \( m \) is even and \( q \) is odd. Let \( \alpha \) be the generator of \( \mathbb{GF}(r)^* \). Let \( \alpha \in \mathbb{GF}_h^{(2, r)} \). It then follows from (12) and Lemma 11 that for any \( \beta \in \mathbb{GF}(r)^* \) the Hamming weight of any codeword

\[ c(\beta) = (\text{Tr}_{r/q}(\beta), \text{Tr}_{r/q}(\beta\theta), \ldots, \text{Tr}_{r/q}(\beta\theta^{n-1})) \]

of the code \( C(r, N) \) is equal to

\[ n - \frac{Z(r, \beta) - 1}{N} = \frac{(q - 1)(r - \sqrt{T})}{qN} > 0. \]

Note that \( |c_1^{(2, r)}| = |c_1^{(2, r)}| = (r - 1)/2 \). The weight distribution and dimension of the code follow. This completes the proof. \( \square \)

Theorem 17 is an extension of Theorem 7 in Baumert and McEliece [2]. We list some numerical examples in Table 1.

6. The weight distribution when \( N_1 = \gcd((r - 1)/(q - 1), N) = 3 \)

Theorem 18. Let \( N \) be a divisor of \( r - 1 \). When \( N_1 = \gcd((r - 1)/(q - 1), N) = 3 \) and \( p \equiv 1 \pmod{3} \), the set \( C(r, N) \) in (1) is a \( [(q^m - 1)/N, m] \) code with the following weight distribution:

\[ 1 + \frac{r - 1}{3} x \frac{(q - 1)(r - 1) c_1 + d_1}{q^m} + \frac{r - 1}{3} x \frac{(q - 1)(r - 1) c_1 + d_1}{q^m} + \frac{r - 1}{3} x \frac{(q - 1)(r - 1) c_1 + d_1}{q^m}, \]

where \( c_1 \) and \( d_1 \) are uniquely given by \( 4q^{m/3} = c_1^2 + 27d_1^2 \), \( c_1 \equiv 1 \pmod{3} \) and \( \gcd(c_1, p) = 1 \).
Proof. By assumption $\gcd(m, q - 1) = 3$. It then follows from (8) that
\[
\left\{ xy : y \in \operatorname{GF}(q)^*, x \in C_i^{(N, r)} \right\} = \frac{3(q - 1)}{N} \ast C_i^{(3, r)}.
\]

Since $N_1 = \gcd((r - 1)/(q - 1), N) = 3$, $(r - 1)/(q - 1) \mod 3 = m \mod 3 = 0$. Note that every element of $\operatorname{GF}(q)^*$ is of the form $\omega^i r^{(r - 1)/(q - 1)}$ for some integer $i$. Hence, $\operatorname{GF}(q)^* \subset C_0^{(3, r)}$. It then follows from Lemma 8 that the Gaussian periods $\eta_i^{(3, r)}$ take only the following three distinct values:
\[
\frac{1 + c_1 r^{1/3}}{3}, \quad \frac{1 - \frac{1}{2}(c_1 + 9d_1)r^{1/3}}{3}, \quad \frac{1 - \frac{1}{2}(c_1 - 9d_1)r^{1/3}}{3}.
\]

It then follows from (12) that for any $\beta \in \operatorname{GF}(r)^*$ the Hamming weight of any codeword
\[
c(\beta) = (\text{Tr}_{r/q}(\beta), \text{Tr}_{r/q}(\beta\theta), \ldots, \text{Tr}_{r/q}(\beta\theta^{n-1}))
\]
of the code $C(r, q - 1)$ is equal to
\[
n - \frac{Z(r, \beta) - 1}{N} = \frac{1}{q} \left[ q + r - 1 + 3(q - 1)\eta_i^{(3, r)} \right] \geq 0.
\]

Note that $|C_i^{(3, r)}| = (r - 1)/3$. The weight distribution and dimension of the code then follow. This completes the proof. \qed

Theorem 18 of this section is an extension of Theorem 14 in [8] and Theorem 6 in [7]. We list some numeral examples in Table 2.

**Table 2**

<table>
<thead>
<tr>
<th>$q$</th>
<th>$m$</th>
<th>$N$</th>
<th>$\gcd \left( \frac{r-1}{3}, N \right)$</th>
<th>$[n, k, d]$</th>
<th>Weight distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>3</td>
<td>6</td>
<td>3</td>
<td>[57, 3, 45]</td>
<td>${1 + 114x^{45} + 114x^{48} + 114x^{54}}$</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>18</td>
<td>3</td>
<td>[19, 3, 15]</td>
<td>${1 + 114x^{15} + 114x^{16} + 114x^{27}}$</td>
</tr>
</tbody>
</table>

**Table 3**

<table>
<thead>
<tr>
<th>$q$</th>
<th>$m$</th>
<th>$N$</th>
<th>$\gcd \left( \frac{r-1}{3}, N \right)$</th>
<th>$[n, k, d]$</th>
<th>Weight distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>6</td>
<td>3</td>
<td>3</td>
<td>[1365, 6, 1008]</td>
<td>${1 + 2730x^{1008} + 1365x^{1056}}$</td>
</tr>
<tr>
<td>6</td>
<td>9</td>
<td>3</td>
<td>[455, 6, 336]</td>
<td>[21, 3, 12]</td>
<td>${1 + 21x^{14} + 42x^{18}}$</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td></td>
<td>[7, 3, 4]</td>
<td>${1 + 21x^{4} + 42x^{6}}$</td>
</tr>
</tbody>
</table>

**Theorem 19.** Let $N$ be a divisor of $r - 1$. Suppose that $\gcd((r - 1)/(q - 1), N) = 3$ and $p \equiv 2 \pmod{3}$. If $s \cdot m \equiv 0 \pmod{4}$, then $C(r, N)$ is a $[(r - 1)/N, m, (q - 1)(r - \sqrt{r})/Nq]$ code over $\operatorname{GF}(q)$ with the weight distribution
\[
1 + \frac{2(r - 1)}{3} x^{\frac{(q-1)(r-2\sqrt{r})}{Nq}} + \frac{r - 1}{3} x^{\frac{(q-1)(r+2\sqrt{r})}{Nq}}.
\]

If $s \cdot m \equiv 2 \pmod{4}$, then $C(r, N)$ is a $[(r - 1)/N, m, (q - 1)(r - 2\sqrt{r})/Nq]$ code over $\operatorname{GF}(q)$ with the weight distribution
\[
1 + \frac{r - 1}{3} x^{\frac{(q-1)(r-2\sqrt{r})}{Nq}} + \frac{2(r - 1)}{3} x^{\frac{(q-1)(r+2\sqrt{r})}{Nq}}.
\]

Proof. Note that $\gcd((r - 1)/(q - 1), N) = 3$ and $p \equiv 2 \pmod{3}$. This theorem becomes a special case of Theorem 23. \qed

We list some numeral examples of Theorem 19 in Table 3.

7. The weight distribution when $N_1 = \gcd((r - 1)/(q - 1), N) = 4$

**Theorem 20.** Let $N$ be a divisor of $r - 1$. If $N_1 = \gcd((r - 1)/(q - 1), N) = 4$ and $p \equiv 1 \pmod{4}$, $C(r, N)$ is a $[(r - 1)/N, m]$ code over $\operatorname{GF}(q)$ with the weight distribution
\[
1 + \frac{r - 1}{4} x^{\frac{(q-1)(r+\sqrt{r}+2u_1)}{Nq}} + \frac{r - 1}{4} x^{\frac{(q-1)(r-\sqrt{r}+4v_1)}{Nq}} + \frac{r - 1}{4} x^{\frac{(q-1)(r+\sqrt{r}-4u_1)}{Nq}} + \frac{r - 1}{4} x^{\frac{(q-1)(r-\sqrt{r}-4v_1)}{Nq}}
\]

where $u_1$ and $v_1$ are given by $q^{m/2} = u_1^2 + 4v_1^2$, $u_1 \equiv 1 \pmod{4}$, and $\gcd(u_1, p) = 1$. 

Tables 3 and 4, where \(N\) formula with the help of Lemma 10 and (12).

**Proof.**

(a) Theorem 21 is an extension of the main results obtained by Baumert and Mykkeltveit [4] and the main results of [5, Section 11.7].

(b) According to the conclusions of [31], there are six subcases for Gauss sums in the index 2 case. Theorem 21 is the corresponding result for one of the six subcases.

Example 3. Let \(q = 2, m = 42\) and \(N = 7^2 = 49\). Then the set \(\mathcal{C}(r, N)\) in (1) is an \([n = 8975051247, k = 42, d = 44877307904]\) code over \(GF(2)\) with the weight distribution

\[
1 + nx^{44877307904} + 3nx^{44877832192} + 21nx^{44877979648} + 21nx^{4487808144} + 3nx^{44878356480}.
\]

Example 4. Let \(q = 3, m = 55\) and \(N = 11^2 = 121\). Then the set \(\mathcal{C}(r, N)\) in (1) is an

\([n = 1441729016604299000588186, k = 55, d = 96115267733830625644778]\) code over \(GF(3)\) with the weight distribution

\[
1 + 6nx^{96115267733830625644778} + 55nx^{96115267735964537698190} + 55nx^{96115267736445713945528} + 5nx^{96115267738914357301436}.
\]
9. The weight distribution in the case that \( n \) is a prime power

The following result is presented in [25].

**Theorem 22.** Let \( q = p^s \). Let \( t \) be an odd prime and \( \ell \) be a positive integer. Assume that the multiplicative order of \( q \) modulo \( t^\ell \) is \( t^d \), where \( 0 \leq d < \ell \). Define \( m = t^d \) and \( N = (q^m - 1)/t^j \) for any \( j \) with \( 1 \leq j \leq \ell \).

If \( j \leq \ell - d \), then the set \( \mathcal{C}(r, N) \) in (1) is a \([t^j, 1, t^j] \) constant-weight code over \( \text{GF}(q) \) with the weight enumerator

\[
1 + (q - 1)x^d.
\]

If \( j > \ell - d \), then the set \( \mathcal{C}(r, N) \) in (1) is a \([t^j, t^{-(\ell-d)}] \) cyclic code over \( \text{GF}(q) \) with the weight enumerator

\[
\sum_{w=0}^{r-\ell+d} \binom{t^j-\ell+d}{w} x^{t^{-(\ell-d)} w}.
\]

**Example 5.** Let \( q = 2^2 \) and \( t^\ell = 3^2 \). Then the order of \( q \) modulo \( t^\ell \) is \( 3^2 \). Define \( m = 3^2 = 9 \) and \( N = (q^m - 1)/t^2 \). Then \( n = t^2 = 9 \), and the set \( \mathcal{C}(r, N) \) in (1) is a \([9, 3, 3] \) cyclic code over \( \text{GF}(4) \) with the weight enumerator

\[
1 + 9x^3 + 27x^6 + 27x^9.
\]

10. The weight distribution in the semi-primitive and related cases

**Theorem 23.** Let \( p \) be a prime and \( s \cdot m \) be even. Let \( N \) be a positive divisor of \( r - 1 \) and \( N_1 = \gcd((r - 1)/(q - 1), N) > 2 \). Assume there exists a positive integer \( j \) such that \( p^j \equiv -1 \pmod{N_1} \), and the \( j \) is the least such. Define \( \gamma = s \cdot m/2 \).

(a) If \( \gamma \cdot p \) and \((p^j + 1)/N_1\) are all odd, then the set \( \mathcal{C}(r, N) \) in (1) is a \([q^m - 1]/N, m \) code over \( \text{GF}(q) \) with the weight enumerator

\[
1 + \frac{r - 1}{N_1} x^{\frac{(q-1)(r-1)\gamma}{q^m}} + \frac{(r - 1)(N_1 - 1)}{N_1} x^{\frac{(q-1)(r-1)\gamma}{q^m}},
\]

provided that \( N_1 < \sqrt{r} + 1 \).

(b) In all other cases, the set \( \mathcal{C}(r, N) \) in (1) is a \([q^m - 1]/N, m \) code with the weight enumerator

\[
1 + \frac{r - 1}{N_1} x^{\frac{(q-1)(r-1)\gamma}{q^m}} + \frac{(r - 1)(N_1 - 1)}{N_1} x^{\frac{(q-1)(r-1)\gamma}{q^m}},
\]

provided that \( \sqrt{r} + (-1)^\gamma (N_1 - 1) > 0 \).

**Proof.** The conclusions of this theorem follow from (12). Lemma 12 and the conditions stated in this theorem. \( \Box \)

Regarding Theorem 23, we have the following remarks.

(a) When \( N_1 = N \), this is the classical semi-primitive case, and the weight distribution of the code was studied by Delsarte and Goethals [6], McEliece [18], and Baumert and McEliece [2].

(b) When \( N_1 < N \), this may not be the semiprimitive case for \( N \). For example, let \( q = 7, m = 2 \) and \( N = 12 \). We now prove that this is not the semi-primitive case for \( N = 12 \). To this end, we prove that there is no positive integer \( j \) such that \( 7^j \equiv -1 \pmod{12} \), which is equivalent to the following system of congruences:

\[
7^j \equiv -1 \pmod{4} \quad \text{and} \quad 7^j \equiv -1 \pmod{3}
\]

by the Chinese Remainder Theorem. The second congruence does not have a solution. In this case \( N_1 = 4 \) and \( N_1 \) divides \( 7^j + 1 \). By Theorem 23 the code over \( \text{GF}(7) \) has length 4, dimension 2 and weight enumerator

\[
1 + 12x^2 + 36x^4.
\]

This shows that some non-semiprimitive cases can be settled using the results of the semiprimitive cases.

(c) The condition that \( N_1 < \sqrt{r} + 1 \) or \( \sqrt{r} + (-1)^\gamma (N_1 - 1) > 0 \) is to ensure that the dimension of the code is \( m \).

(d) Theorem 2.1 in [9] is a special case of Theorem 23 above.

Theorem 23 describes a class of two-weight irreducible cyclic codes over \( \text{GF}(q) \), and is an extension of Theorem 6 in Baumert and McEliece [2]. It is an interesting problem to find out all two-weight irreducible cyclic codes over \( \text{GF}(q) \). Schmidt and White have given a characterization of all two-weight irreducible cyclic codes over \( \text{GF}(q) \) when \( q \) is prime [23], and Rao and Pinnawala [22] have constructed another type of two-weight codes. However, the conditions for the characterization given in [23] may not be easily used for finding out all two-weight irreducible cyclic codes over \( \text{GF}(p) \). It follows from (10) that the code \( \mathcal{C}(r, N) \) in (1) has at most two nonzero weights if and only if the Gaussian periods \( \eta_{\gcd((r-1)/(q-1),N),r} \) take on at most two distinct values. A special case of this is the case of uniform cyclotomy [3]. It might be possible to give another characterization in this direction.
11. The weight distribution in a few other cases and other results

Gaussian periods of order 5, 6, 8 and 12 are computed in [13] and [11] respectively. So the weight distribution of the code $C(r, N)$ in (1) can be computed by these Gaussian periods and (12). However, the weight formulas will be complicated due to the messy expression of these Gaussian periods. Two-weight projective irreducible cyclic codes are characterized by Wolfmann [30].

Two recursive algorithms were developed for computing the weight distribution of certain irreducible cyclic codes [20]. The weight enumerators of all nondegenerate irreducible cyclic binary $[n, m]$-codes have been computed for which $N = (2^m - 1)/n < 500, k \leq 27$ by MacWilliams and Seery [16], and $k > 27$ by Ward [29]. The weights of irreducible cyclic codes are discussed by Aubry and Langevin [1], Moisio [19] and by Segal and Ward [24]. The relations between the weight distributions of irreducible cyclic codes and the Hasse–Davenport curves are dealt with by van der Vlugt [27]. Chains of irreducible cyclic codes and relations among their weight distributions are presented in [14, 12].

12. Bounds on weights in irreducible cyclic codes

Since it is notoriously hard to determine the weight distributions of the irreducible cyclic codes, it would be interesting to develop tight bounds on the weights in irreducible cyclic codes. Such tight bounds can give information on the error-correcting capability of this class of cyclic codes. The objective of this section is to develop such tight bounds.

**Theorem 24.** Let $N$ be a positive divisor of $r - 1$ and define $N_1 = \gcd((r - 1)/(q - 1), N)$. Let $m_0$ be the multiplicative order of $q \mod n$. Then the set $C(r, N)$ in (1) is a $[q^{m_0} - 1]/N, m_0] \text{ cyclic code over GF}(q)$ in which the weight $w$ of every nonzero codeword satisfies that

$$w_H(c(\beta)) \geq (q - 1) \left\lfloor \frac{r - \lfloor (N_1 - 1)\sqrt{r} \rfloor}{qN} \right\rfloor,$$

$$w_H(c(\beta)) \leq (q - 1) \left\lceil \frac{r + \lfloor (N_1 - 1)\sqrt{r} \rfloor}{qN} \right\rceil.$$

In particular, if $N_1(N_1 - 1) < r$, then $m_0 = m$.

**Proof.** The results of this theorem follow from **Theorem 13** and (12).

The lower bound of **Theorem 24** is tight when $\gcd((r - 1)/(q - 1), N)$ is small, but it may be not in some other cases. In fact, when $\gcd((r - 1)/(q - 1), N) = 1$, the lower and upper bounds of **Theorem 24** are the same, and they are indeed achieved as the code in this case is a constant-weight code. **Table 5** lists some experimental data, where $n, k, d$ are the length, dimension and minimum nonzero weight of the code.

13. Summary and open problems

The contributions of this paper include the following:

- A survey of earlier results on the weight distributions of irreducible cyclic codes.
- Extensions and generalizations of earlier results on the weight distributions of irreducible cyclic codes (**Theorems 23, 21, 15, 17, 18, 19 and 20**).
- A complete characterization of one-weight irreducible cyclic codes (**Theorem 16**), which is an extension of the result in [28].
- The weight divisibility of irreducible cyclic codes (**Theorem 14**).
- A lower and upper bound on the weights in irreducible cyclic codes (**Theorem 24**).
- A property on Gaussian periods (**Theorem 13**).
While it is hard to determine the weight distributions of the irreducible cyclic codes in general, it is possible to solve this problem for other special cases. One open problem would be a simpler characterization of two-weight irreducible cyclic codes than the one presented in [23] by Schmidt and White.

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**References**


